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Generating systems for modular forms for the Weil representation and Hecke operators for orthogonal modular forms

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Zusammenfassung

Das Transformationsverhalten der vektorwertigen Thetafunktion eines positiv-definiten geraden Gitters unter der metaplektischen Gruppe $\mathrm{Mp}_2(\mathbb{Z})$ wird durch die Weil-Darstellung beschrieben. Im ersten Teil dieser Arbeit untersuchen wir Modulformen zur Weil-Darstellung. Dies ist in drei Projekte unterteilt:

Für eine isotrope Untergruppe H einer Diskriminantenform D existiert eine Anhebung von Modulformen für die Weil-Darstellung der Diskriminantenform $H^\perp/H$ zu Modulformen für die Weil-Darstellung von D. Wir bestimmen eine Menge von Diskriminantenformen, sodass alle Modulformen für jede Diskriminantenform von den Diskriminantenformen in dieser Menge induziert sind. Außerdem existieren für jede Diskriminantenform in dieser Menge Modulformen, die nicht von kleineren Diskriminantenformen induziert sind.

Als nächstes untersuchen wir die Invarianten der Weil-Darstellung. Insbesondere zeigen wir, dass diese von 5 fundamentalen Invarianten induziert sind.

Im dritten Projekt zeigen wir, dass der Raum der Spitzenformen für die Weil-Darstellung durch Thetareihen erzeugt wird. Dies gibt eine positive Antwort auf Eichlers Basisproblem in diesem Fall. Als Anwendung leiten wir Waldspurgers Ergebnis zum Basisproblem für skalare Modulformen her.

Der zweite Teil dieser Arbeit befasst sich mit orthogonalen Modulformen. Zunächst geben wir einen neuen Beweis der Surjektivität des multiplikativen Borcherds-Lift, der auf der Analyse der lokalen Picard-Gruppen basiert und sich unmittelbar aus dem Basisproblem ergibt. Dann untersuchen wir orthogonale Hecke-Operatoren, insbesondere berechnen wir die Hecke-Eigenwerte von Borcherds Φ_{12}.

Abstract

The transformation behaviour of the vector-valued theta function of a positive-definite even lattice under the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ is described by the Weil representation. In the first part of this thesis we study modular forms for the Weil representation. This is divided into three projects:

For an isotropic subgroup H of a discriminant form D there exists a lift from modular forms for the Weil representation of the discriminant form $H^{\perp}/H$ to modular forms for the Weil representation of D. We determine a set of discriminant forms such that all modular forms for any discriminant form are induced from the discriminant forms in this set. Furthermore, for any discriminant form in this set there exist modular forms that are not induced from smaller discriminant forms.

Next we investigate the invariants of the Weil representation. In particular, we show that they are induced from 5 fundamental invariants.

In the third project we show that the space of cusp forms for the Weil representation is generated by theta series. This gives a positive answer to Eichler's basis problem in this case. As application we derive Waldspurger's result on the basis problem for scalar-valued modular forms.

The second part of this thesis is about orthogonal modular forms. First, we give a new proof of the surjectivity of the multiplicative Borcherds lift based on the analysis of local Picard groups that follows immediately from the basis problem. Then we study orthogonal Hecke operators, in particular, we compute the Hecke eigenvalues of Borcherds' Φ_{12}.

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The first part of this thesis is based on the articles [52], [53] and [51], where [53] is joint work with Nils Scheithauer. The last two chapters are based on joint work with Nils Scheithauer and Moritz Dittmann.

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Contents

Introduction

Modular forms, which are the object of study of this thesis, have numerous applications in both number theory and geometry. They arise naturally in various branches of mathematics, uncovering profound connections between different areas.

Let $\mathcal{H}$ be a hermitian symmetric space, Γ a group that acts properly discontinuously on $\mathcal{H}$ from the left and $\rho : \Gamma \to \mathrm{GL}(V)$ a complex representation of Γ. A modular form of weight k for the representation ρ of Γ is a holomorphic (or sometimes meromorphic) function $f : \mathcal{H} \to V$ that satisfies

$$f(\gamma z) = j(\gamma, z)^k \rho(\gamma) f(z)$$

for all $z \in \mathcal{H}$, $\gamma \in \Gamma$ and some slow growth condition at the boundary, where $j : \Gamma \times \mathcal{H} \to \mathbb{C}$ is a factor of automorphy. Often $\mathcal{H}$ will be given by $G(\mathbb{R})/K$, where G is a $\mathbb{Z}$-group scheme and $K \subset G(\mathbb{R})$ is a compact subgroup. Then $G(\mathbb{R})$ naturally acts on $\mathcal{H}$ and Γ will be some subgroup of $G(\mathbb{Q})$ commensurable with $G(\mathbb{Z})$.

Consider for example $G = \mathrm{SL}_2$. Let $\mathbb{H} \coloneqq \{\tau = x + iy \in \mathbb{C} \mid y > 0\}$ be the complex upper half-plane. The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ from the left via Möbius transformation $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \tau = \frac{a\tau + b}{c\tau + d}$. Then $\mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}(2)$ and $\mathbb{H}$ are biholomorphic under

$$\gamma \, \mathrm{SO}(2) \mapsto \gamma i$$

and $j(\gamma, \tau) \coloneqq c\tau + d$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ is a factor of automorphy. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be some congruence subgroup and $\chi : \Gamma \to \mathbb{C}$ a character. Modular forms of weight k and character χ for Γ are the classical modular forms that have been studied first. They play a central role, for example, in the famous modularity theorem, which roughly states that the points over $\mathbb{F}_p$ of a rational elliptic curve of conductor N are counted by a modular form or more precisely a cuspform of weight 2 for $\Gamma_0(N)$.

In this thesis we will study modular forms for the Weil representation of $\mathrm{Mp}_2(\mathbb{Z})$ as well as modular forms for orthogonal groups.

Modular forms for the Weil representation

The Weil representation describes the transformation behavior of the vector-valued theta series of an even lattice under the group $\mathrm{Mp}_2(\mathbb{Z})$ (see below). It is a special case of the representations of symplectic groups constructed by Weil in [69] and plays an important role in representation theory, the theory of modular forms and even quantum field theory.

A discriminant form is a finite abelian group D with a non-degenerate quadratic form $\mathrm{q} : D \to \mathbb{Q}/\mathbb{Z}$. The level of D is the smallest positive integer N such that $N\,\mathrm{q}(\gamma) \in \mathbb{Z}$ for all $\gamma \in D$.

The metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ is a double cover of $\mathrm{SL}_2(\mathbb{Z})$. Its elements can be described as the pairs (M, ϕ), where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\phi : \mathbb{H} \to \mathbb{C}$ is a holomorphic function such that $\phi(\tau)^2 = j(M, \tau) = c\tau + d$.

For every discriminant form D, there exists a representation $\rho_D : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C}[D])$ called the Weil representation, which we describe in detail in Chapter 1. A function $f : \mathbb{H} \to \mathbb{C}[D]$ is called a modular form of weight $k \in \frac{1}{2}\mathbb{Z}$ for the Weil representation if (cf. Definition 1.3.1)

(i) $f(M\tau) = \phi(\tau)^{2k} \rho_D((M, \phi)) f(\tau)$ for all $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$

(ii) f is holomorphic

(iii) f is bounded as $\tau \to i\infty$.

We denote the space of modular forms of weight k for the Weil representation by $\mathrm{M}_k(D)$. Furthermore, we denote by $\mathrm{S}_k(D)$ the space of cusp forms of weight k for ρ_D, which are the modular forms that vanish at infinity. In the first part of this thesis we study these types of modular forms.

A first example of modular forms for the Weil representation are theta series. Let L be a positive-definite even lattice, i.e. a free $\mathbb{Z}$-module of finite rank m with a non-degenerate symmetric bilinear form $(\cdot, \cdot)$ such that $(\lambda, \lambda) \in 2\mathbb{Z}_{\geq 0}$ for all $\lambda \in L$. We denote by L' the dual lattice of L. Then L'/L is a discriminant form with quadratic form given by $\mathrm{q}(\gamma) = (\gamma, \gamma)/2 \bmod 1$. Let $P \in \mathbb{C}[x_1, \ldots, x_m]$ be a harmonic polynomial of homogeneous degree $k - m/2$ with $k \geq m/2$. The vector-valued theta series $\theta_{L,P}$ is given by

$$\theta_{L,P}(\tau) \coloneqq \sum_{\lambda \in L'} P(\lambda) e^{\pi i (\lambda, \lambda) \tau} e^{\lambda + L} = \sum_{\gamma \in L'/L} \theta_{\gamma,P}(\tau) e^{\gamma},$$

where $\theta_{\gamma,P}(\tau) = \sum_{\lambda \in \gamma + L} P(\lambda) e^{\pi i (\lambda, \lambda) \tau}$. By the Poisson summation formula, $\theta_{L,P}$ is a modular form of weight k for the Weil representation corresponding to L'/L.

The signature $\operatorname{sign}(D) \in \mathbb{Z}/8\mathbb{Z}$ of a discriminant form D is given by the signature of any even lattice with that discriminant form. The non-trivial element in the kernel of the covering map $\mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$ acts as $(-1)^{\operatorname{sign}(D)}$ so that the Weil representation ρ_D descends to a representation of $\mathrm{SL}_2(\mathbb{Z})$ if D has even signature.

We now describe the individual main results in detail.

Isotropic Lifts

Let H be an isotropic subgroup of a discriminant form D. Then $H^\perp/H$ is a discriminant form of the same signature as D and of order $|H^\perp/H| = |D|/|H|^2$. There is an isotropic lift $\uparrow_H := \uparrow_H^D \colon \mathbb{C}[H^\perp/H] \to \mathbb{C}[D]$ which commutes with the corresponding Weil representations (see Section 2.1). In particular, $\uparrow_H$ maps modular forms to modular forms. The modular forms of the form $\uparrow_H(f)$ for $f \in \mathrm{M}_k(H^\perp/H)$ span a subspace of $\mathrm{M}_k(D)$. Since small discriminant forms are much easier to understand than large ones, it is important to know when this subspace is all of $\mathrm{M}_k(D)$.

In Chapter 2 we determine a set of discriminant forms such that for any discriminant form D the space $\mathrm{M}_k(D)$ is generated by modular forms of the form $\uparrow_H(f)$, where $H^\perp/H$ is isomorphic to a discriminant form in this set. Furthermore, this set is minimal in the sense that for any such discriminant form, there exists a modular form that is not induced from any smaller discriminant form. We call the discriminant forms in this set of small type:

First assume that D is a discriminant form of level a power of a prime p. If p is odd we say that D is of small type if one of the following conditions holds:

(i) D has rank two or less.

(ii) D has rank three and at least one Jordan component is of level p.

(iii) D has rank four and is of type $p^{-\epsilon 2} q_1^{\pm 1} q_2^{\pm 1}$, where $\epsilon = \left(\frac{-1}{p}\right)$ and q_1, q_2 are powers of p and can also be p.

(iv) D has rank five and is of level p.

Here we have used the notation of Conway and Sloane for the description of discriminant forms (cf. [20, Chapter 15] and Section 1.1).

If $p = 2$, there is a similar characterization (cf. Section 2.4). If D has level N and $N = \prod_{p|N} p^{\nu_p}$ is the prime decomposition of N, then D decomposes into the orthogonal sum of its p-subgroups

$$D = \bigoplus_{p|N} D_{p^{\nu_p}}$$

where $D_{p^{\nu_p}}$ is the kernel of $\gamma \mapsto p^{\nu_p} \cdot \gamma$. We say that a discriminant form of level N is of small type if for all $p \mid N$ the p-subgroups $D_{p^{\nu_p}}$ of D are of small type. We remark that any discriminant form of rank ≥ 7 is not of small type and for a fixed level there are only finitely many discriminant forms of small type. Now the main result of Chapter 2 is Theorem 2.4.1:

Let D be a discriminant form. Then all modular forms for the Weil representation of D are linear combinations of modular forms of the form $\uparrow_H (f)$, where $H \subset D$ is an isotropic subgroup such that $H^\perp/H$ is of small type and f is a modular form for the Weil representation of $H^\perp/H$. For any discriminant form of small type there exist modular forms which are not linear combinations of isotropically lifted modular forms.

This result is one ingredient for a theory of vector-valued newforms for the Weil representation, which is still to be developed (cf. [12]).

We sketch the proof of the theorem. We say that a modular form $f \in \mathrm{M}_k(D)$ is a linear combination of isotropically lifted modular forms if we can write

$$f = \sum_{0 \neq H} \uparrow_H (f_H)$$

for suitable modular forms $f_H \in \mathrm{M}_k(H^\perp/H)$. Here we sum over all non-trivial isotropic subgroups. We show that for a discriminant form D all modular forms are linear combinations of isotropically lifted modular forms if and only if D is not of small type. Using the fact that the lifts are transitive, Theorem 2.4.1 then follows by induction on the order of D. We describe the first step in more detail.

We show that for a modular form f being a linear combination of isotropically lifted modular forms is actually a pointwise property: f is a linear combination of isotropically lifted modular forms if and only if for every τ the point $f(\tau) \in \mathbb{C}[D]$ is a linear combination of isotropically lifted points $v \in \mathbb{C}[H^\perp/H]$. One direction of this equality is trivial, the other one is proved in Proposition 2.2.2. For k large enough every subrepresentation of ρ_D contains some non-trivial modular form of weight k. We deduce that all modular forms for all weights are linear combinations of isotropically lifted modular forms if and only if the space generated by the images of the maps $\uparrow_H$ is all of $\mathbb{C}[D]$ (see Theorem 2.2.4). The latter question can be reduced to the p-subgroups.

So it remains to show that the isotropic lifts generate $\mathbb{C}[D]$ if and only if D is not of small type. The idea of the proof is to find some condition on $\gamma \in D$ which is equivalent to e^γ being a linear combination of isotropic lifts. For p odd this condition says that $\gamma^\perp$ contains some isotropic subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$ (see

Proposition 2.3.7). When D contains an isotropic subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$, then this is the case for all $\gamma \in D$. Interestingly the other direction holds as well for all discriminant forms except for $p^{-\epsilon 6}$ with $\epsilon = \left(\frac{-1}{p}\right)$. This form does not contain an isotropic subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$, but still for every $\gamma \in D$ the subgroup $\gamma^\perp$ contains an isotropic subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$. The form $p^{-\epsilon 6}$ is the only discriminant form with this property.

In the case $p = 2$ the same condition also implies that $e^\gamma \in \mathrm{im}(\uparrow)$, however it is too strong and the other direction does not hold. We will use a result from graph theory for a sharper condition (see Proposition 2.3.11).

In both cases we determine the discriminant forms that do not contain an isotropic subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$ and then check for which of them the condition holds for all $\gamma \in D$ and for which there exists a $\gamma \in D$, where the condition is not met (see Theorems 2.3.10 and 2.3.18). We remark that, if a p-adic discriminant form is of small type, then it does not contain an isotropic subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$. Furthermore, there are very few p-adic discriminant forms that are not of small type and do not contain an isotropic subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$.

This chapter is based on the article [52].

Invariants of the Weil representation

The invariants of the Weil representation of $\mathrm{Mp}_2(\mathbb{Z})$ are the modular forms of weight 0. For several applications it is important to have an explicit description of them. For example the space of Jacobi forms of lattice index L and singular weight is naturally isomorphic to the space of invariants $\mathbb{C}[L'/L]^{\mathrm{Mp}_2(\mathbb{Z})}$ (cf. [66]). In Chapter 3 we investigate the space of invariants for the Weil representation. In particular, we will specialize the main theorem on isotropically lifted modular forms to invariants. We already know that any invariant must be a linear combination of lifted invariants from discriminant forms of small type. Any discriminant form of small type contains modular forms that can not be obtained from lifts. However, we should expect that sometimes these modular forms have positive weight. So when we only consider invariants, then the list of discriminant forms which contain invariants that are not induced from smaller discriminant forms should be much smaller. In fact, we show that all invariants are induced from 5 fundamental invariants. This result generalizes the special case of when the corresponding discriminant form possesses self-dual isotropic subgroups, in which case the invariants are generated by the characteristic functions of these groups (cf. [66], [3] and [54]).

Since the non-trivial element in the kernel of the covering map $\mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$ acts as $(-1)^{\mathrm{sign}(D)}$, the space of invariants $\mathbb{C}[D]^{\mathrm{Mp}_2(\mathbb{Z})}$ is trivial if D has odd signature. Hence, we can restrict to the case that the signature of D is even when we study the subspace of invariants. Recall that in this case the Weil representation ρ_D descends to a representation of $\mathrm{SL}_2(\mathbb{Z})$.

Now let D be a discriminant form of even signature and level N. Then the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ factors through the finite group $\Gamma(N)\backslash\mathrm{SL}_2(\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Hence, we can project onto the subspace of invariants by averaging. We define the map

$$\mathrm{inv}_D : \mathbb{C}[D] \to \mathbb{C}[D]$$

by

$$\mathrm{inv}_D(e^\gamma) = \frac{1}{|\Gamma(N)\backslash\mathrm{SL}_2(\mathbb{Z})|} \sum_{M\in\Gamma(N)\backslash\mathrm{SL}_2(\mathbb{Z})} \rho_D(M^{-1})e^\gamma.$$

It maps onto the subspace of invariants $\mathbb{C}[D]^{\mathrm{SL}_2(\mathbb{Z})}$. Using the explicit formulas for the Weil representation given in [60] we can determine an explicit formula for inv_D and derive a simple dimension formula for the subspace of invariants (see Theorem 3.1.2). We compute the formulas for the projection and the dimension explicitly for discriminant forms of prime level (see Section 3.2).

Let $N = \prod_{p|N} p^{\nu_p}$ be the prime decomposition of N and recall that D decomposes into an orthogonal sum of its p-subgroups $D_{p^{\nu_p}}$. We find that

$$\mathbb{C}[D]^{\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})} \cong \bigotimes_{p|N} \mathbb{C}[D_{p^{\nu_p}}]^{\mathrm{SL}_2(\mathbb{Z}/p^{\nu_p}\mathbb{Z})}$$

so that in order to describe the invariants of the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ it suffices to consider p-adic discriminant forms.

For this purpose we define 5 fundamental discriminant forms $D_p^{x,s}$ of square class x and signature s. Using the above formula for the projection we show that their subspace of invariants is 1-dimensional and we determine a generator $i_p^{x,s}$. We list them in the following tables. For odd p they are given by

$D_p^{x,s}$	square class	signature	$i_p^{x,s}$
0	square	$0 \bmod 8$	e^0
p^{-4}	square	$4 \bmod 8$	$(p-1)e^0 - \sum_{\gamma\in M} e^\gamma$
$p^{\epsilon 3}$	non-square	$0 \bmod 2$	$\sum_{\gamma\in M^+} e^\gamma - \sum_{\gamma\in M^-} e^\gamma$

and for $p = 2$ by

$D_p^{x,s}$	square class	signature	$i_p^{x,s}$
0	square	$0 \mod 8$	e^0
2_{II}^{-4}	square	$4 \mod 8$	$e^0 - \sum_{\gamma\in M} e^\gamma$
$2_t^{+2}4_{II}^{+2}$	square	$t = 2 \mod 4$	$\sum_{\gamma\in M^+} e^\gamma - \sum_{\gamma\in M^-} e^\gamma$
$2_1^{+1}4_t^{\epsilon}\,8_{II}^{+2}$	non-square	$1+t = 0 \mod 2$	$\sum_{\gamma\in M^+} e^\gamma - \sum_{\gamma\in M^-} e^\gamma$

Here we wrote M for the set of isotropic elements whose order is equal to the level of $D_p^{x,s}$ and remark that M has a canonical decomposition $M = M^+ \cup M^-$ in the indicated cases. The main result of Chapter 3 is the following (see Theorem 3.4.6):

Let D be a discriminant form of even signature s, square class x and level p^l, where p is a prime. Then the invariants of the Weil representation on $\mathbb{C}[D]$ are generated by the invariants $\uparrow_H^D (i_p^{x,s})$, where H is an isotropic subgroup of D such that $H^\perp/H$ is isomorphic to the discriminant form $D_p^{x,s}$.

We remark that Skoruppa's result in [66] corresponds to the case that $D_p^{x,s}$ is trivial.

The idea of the proof is similar to that of Theorem 2.4.1. We show that for each $\gamma \in D$, $\mathrm{inv}_D(e^\gamma)$ is a linear combination of isotropic lifts of invariants for suitable isotropic subgroups unless $\mathrm{inv}_D(e^\gamma) = 0$ or D is the fundamental discriminant form $D_p^{x,s}$. Then by induction on the order of D, $\mathrm{inv}_D(e^\gamma)$ is a linear combination of compositions of isotropic lifts of invariants of the form $i_p^{x,s}$. The statement now follows from the transitivity of the isotropic lift.

As an application of the main result we show (see Theorem 3.5.2):

Let L be a positive-definite even lattice of rank n and level N. Suppose n is even. For $p\,|\,N$ we denote the square class and the signature of the p-adic component of L'/L by x_p resp. s_p. Let $\mathcal{L}$ be the set of all overlattices $M \supset L$ such that the p-adic component of M'/M is isomorphic to $D_p^{x_p,s_p}$ for all $p\,|\,N$. Then

$$J_{n/2,L} = \sum_{M\in\mathcal{L}} \mathbb{C}\Bigg(\sum_{\gamma\in M'/M} v_\gamma \vartheta_{M,\gamma} \Bigg),$$

where $\sum_{\gamma\in M'/M} v_\gamma e^\gamma \in \mathbb{C}[M'/M]^{\mathrm{SL}_2(\mathbb{Z})}$ is the invariant corresponding to the product $\prod_{p\,|\,N} i_p^{x_p,s_p}$.

Here

$$\vartheta_{M,\gamma}(\tau,z)=\sum_{\alpha\in\gamma+M} e(\tau(\alpha,\alpha)/2+(\alpha,z))$$

is the Jacobi theta function of the coset $\gamma+M$.

This chapter is based on joint work with N. Scheithauer [53].

The basis problem for the Weil representation

Let N be square-free and $m=0 \bmod 4$. In [25] Eichler announced and later proved in [26] that the space of newforms of weight $m/2$ for $\Gamma_0(N)$ has a basis consisting of theta series corresponding to lattices of level N. These are scalar-valued theta series, which are the 0-components of the vector-valued theta series defined earlier. In general, finding a basis of an appropriate space of modular forms consisting of theta series is known as the basis problem. For classical modular forms it was mostly solved by Waldspurger in [68]. He proved that for all positive integers $m=0 \bmod 4$ and $k\geq m/2$ the space of newforms of weight k for $\Gamma_0(N)$ and trivial character is generated by theta series of lattices of rank m and level N weighted with harmonic polynomials of degree $k-m/2$. For non-trivial character he proved the result when N is a square free integer congruent to 1 modulo 4. Furthermore, in [5] it was shown that for $k>2n+1$ the space of Siegel cusp forms of weight k and genus n for $\Gamma_0(N)$ with N square free is generated by harmonic theta series of suitable lattices.

Two positive-definite even lattices L and M of rank m have isomorphic discriminant forms D and hence isomorphic Weil representations if and only if they are in the same genus, which is denoted by $II_{m,0}(D)$. It is a natural question whether the space of cusp forms $\mathrm{S}_k(D)$ is generated by the vector-valued theta series in the genus $II_{m,0}(D)$. In Chapter 4 we answer this question in the affirmative if the rank of the lattice is sufficiently large compared to the p-ranks of the discriminant form. Because there is no canonical isomorphism between discriminant forms of lattices in the same genus, there is also no canonical way to identify their Weil representations. Therefore, we define for a discriminant form D

$$\begin{aligned}\Theta_{m,k}(D) \coloneqq \operatorname{span}\{\sigma^*\theta_{L,P} \mid L\in II_{m,0}(D),\\ P \text{ harmonic of degree } k-m/2,\ \sigma\in\operatorname{Iso}(D,L'/L)\},\end{aligned}$$

where $\sigma^*\theta_{L,P}=\sum_{\gamma\in D}\theta_{\sigma(\gamma),P}e^{\gamma}$. The main result of Chapter 4 is Theorem 4.3.7:

Let D be a discriminant form of even signature $\operatorname{sign}(D)$ *and m a positive integer such that $m=\operatorname{sign}(D) \bmod 8$, $m>p$-rank(D) for all primes p and $m>6$. Then there are positive-definite even lattices L such that $L'/L\cong D$, i.e. the genus $II_{m,0}(D)$*

is non-empty. Suppose for any L of genus $II_{m,0}(D)$ the $\mathbb{Z}_p$-lattice $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ splits a hyperbolic plane over $\mathbb{Z}_p$ for all primes p. Then

$$\mathrm{S}_k(D) \subset \Theta_{m,k}(D)$$

for all $k \geq m/2$.

We remark that a p-adic lattice L_p of rank m splits a hyperbolic plane over $\mathbb{Z}_p$ if and only if p-rank$(D) < m-2$ or p-rank$(D) = m-2$ and $\prod_q \epsilon_q = \left(\frac{-a}{p}\right)$, where the p-adic component of D is equal to

$$\bigoplus_q q^{\epsilon_q n_q}$$

in the notation of Conway and Sloane (cf. [20], chapter 15) and $|D| = p^\alpha a$ with $(a,p) = 1$. This is a property of the genus of L, rather than of L itself.

For simplicity, we restrict to the case m even. It is likely possible to extend the result to m odd. Then we have to consider the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ instead of $\mathrm{SL}_2(\mathbb{Z})$. Also the condition $m > 6$ can possibly be relaxed. Then however, we will need to deal with some issues regarding convergence of some of the objects used in the proof.

The proof of Theorem 4.3.7 uses the so-called doubling method that was also employed in [5] and [28]. In what follows we give a short description of the proof idea.

First let us consider the case $k = m/2$. The lattices contributing to $\Theta_{m,k}(D)$ are in the genus $II_{m,0}(D)$, which we will denote by G. We define a map $\Phi_D : \mathrm{S}_k(D) \to \Theta_{m,k}(D)$ by

$$\Phi_D(f) \coloneqq \mu(G)^{-1} \cdot |\,\mathrm{O}(D)|^{-1} \cdot \sum_{L \in G} \frac{1}{\#\,\mathrm{Aut}(L)} \sum_{\sigma \in \mathrm{Iso}(D, L'/L)} (f, \sigma^* \theta_L) \cdot \sigma^* \theta_L.$$

Here the first sum ranges over the positive-definite even lattices in G, $\mu(G)$ denotes the mass of G, $(\cdot,\cdot)$ the Petersson scalar product and $\mathrm{O}(D) = \mathrm{Iso}(D,D)$ the orthogonal group of D. The map Φ_D sends cusp forms to cusp forms with image $\Theta_{m,k}(D) \cap \mathrm{S}_k(D)$. We want to show that it is injective. The definition of vector-valued theta series can be extended to Siegel theta series of genus 2. We find that

$$\Phi_D(f)(z') = \int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} \langle f(z), \theta_G^{(2)}\left(\left(\begin{smallmatrix} z & 0 \\ 0 & -\overline{z'} \end{smallmatrix}\right)\right) \rangle y^k \frac{\mathrm{d}x \mathrm{d}y}{y^2},$$

where

$$\theta_G^{(2)} = \mu(G)^{-1} \cdot |\,\mathrm{O}(D)|^{-1} \cdot \sum_{L \in G} \frac{1}{\#\,\mathrm{Aut}(L)} \sum_{\sigma \in \mathrm{Iso}(D, L'/L)} \sigma^* \theta_L^{(2)}$$

is the Siegel genus theta series. By the Siegel–Weil formula, $\theta_G^{(2)}$ is equal to the Siegel Eisenstein series $E_{k,D}^{(2)}$. Studying $E_{k,D}^{(2)}\left(\left(\begin{smallmatrix} z & 0 \\ 0 & z' \end{smallmatrix}\right)\right)$ we show that Φ_D can be expressed in terms of Hecke operators for the Weil representation, i.e.

$$\Phi_D = C(k)\frac{e(-\operatorname{sign}(D)/8)}{\sqrt{|D|}}\sum_{l=1}^{\infty}\frac{T(l^2)}{l^{2k-2}},$$

where $C(k)$ is some non-zero constant depending only on k. We will see that if $\Phi_{L'/L}(f) = 0$, then for any sublattice $M \subset L$ we also have $\Phi_{M'/M}(\uparrow_{L/M}^{M'/M}(f)) = 0$. On the other hand, by investigating the action of the Hecke operators for primes dividing the level of D, we find that $\Phi_D(f) = 0$ implies that f has certain symmetries. Finally, we show that if the conditions of Theorem 4.3.7 are satisfied, then we find a sublattice $M \subset L$ such that $\uparrow_{L/M}^{M'/M}(f)$ has the required symmetry only if $f = 0$. Thus, Φ_D is injective.

For $k = m/2 + h$ with $h > 0$ we apply a certain differential operator ∂_h to the genus theta series and then proceed analogously to the case $k = m/2$.

As application of our main theorem we show how Waldspurger's result can be derived from it.

This chapter is based on the preprint [51].

Modular forms for orthogonal groups

The second part of this thesis is concerned with orthogonal modular forms: Let L be an even lattice of signature $(n, 2)$ with $n \geq 3$. We denote by $\mathrm{O}(L)^+$ the elements in the orthogonal group of L with positive spinor norm. It acts properly discontinuously on the hermitian symmetric space $\mathrm{O}(L \otimes_{\mathbb{Z}} \mathbb{R})/(\mathrm{O}(n) \times \mathrm{O}(2))$. The complex manifold

$$\mathcal{K} = \{[Z_L] \in \mathbb{P}(L \otimes_{\mathbb{Z}} \mathbb{C}) \mid (Z_L, Z_L) = 0, (Z_L, \overline{Z_L}) < 0\}$$

has two connected components. We choose one of them and denote it by $\mathcal{K}^+$. Then $\mathcal{K}^+$ and $\mathrm{O}(L \otimes_{\mathbb{Z}} \mathbb{R})/(\mathrm{O}(n) \times \mathrm{O}(2))$ are biholomorphic.

We can write $L \otimes_{\mathbb{Z}} \mathbb{Q} = (K \oplus II_{1,1}) \otimes_{\mathbb{Z}} \mathbb{Q}$, where K is an even lattice of signature $(n-1, 1)$. Then

$$\mathcal{H}^{\pm} = \{X + iY \in K \otimes_{\mathbb{Z}} \mathbb{C} \mid (Y, Y) < 0\}$$

is biholomorphic to $\mathcal{K}$. We denote the component that is mapped to $\mathcal{K}^+$ by $\mathcal{H}$.

Let $\Gamma \subset \mathrm{O}(L)^+$ have finite index and let $\chi : \Gamma \to \mathbb{C}$ be a character. A meromorphic function $\psi : \mathcal{H} \to \mathbb{C}$ is called an orthogonal modular form of weight k and character χ for Γ if

$$\psi(\gamma Z) = j(\gamma, Z)^k \chi(\gamma)\psi(Z)$$

for all $\gamma \in \Gamma$ and $Z \in \mathcal{H}$ and ψ is meromorphic at the boundary, where $j(\gamma, Z)$ is a given factor of automorphy (cf. Section 5.2). If ψ is actually holomorphic on $\mathcal{H}$ and at the boundary, then we say that ψ is a holomorphic modular form. We denote the space of holomorphic orthogonal modular forms of weight k and character χ for Γ by $\mathcal{M}_k(\Gamma, \chi)$.

The boundary of $\mathcal{K}^+$ in $\mathcal{N} := \{[Z_L] \in \mathbb{P}(L \otimes_{\mathbb{Z}} \mathbb{C}) \mid (Z_L, Z_L) = 0\}$ has 0- and 1-dimensional components, which are in 1-1-correspondence with the 1- and 2-dimensional isotropic subspaces of $L \otimes_{\mathbb{Z}} \mathbb{R}$, respectively. A 1-dimensional boundary component can be identified with a usual complex upper half-plane and the restriction of an orthogonal modular form to a rational 1-dimensional boundary component is a modular form for a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. The operator that projects a modular form to a rational boundary component is called the Siegel Φ-operator.

The complex manifold $\Gamma\backslash\mathcal{K}^+$ can be compactified by adding finitely many 0- and 1-dimensional cusps, which are the Γ-orbits of the rational boundary components of $\mathcal{K}^+$. This compactification is called the Bailey-Borel compactification of $\Gamma\backslash\mathcal{K}^+$ and we denote it by X_Γ.

Orthogonal modular forms and Borcherds products

In [7] Borcherds constructed a lift from modular forms of weight $1 - n/2$ for the Weil representation of $\mathrm{Mp}_2(\mathbb{Z})$ to meromorphic orthogonal modular forms for finite index subgroups Γ of $\mathrm{O}(L)^+$ (see [7, Theorem 13.3]). These orthogonal modular forms have product expansions at 0-dimensional cusps and are therefore called automorphic products or Borcherds products. The divisor of a Borcherds product, which is the formal sum of its zeros and poles with multiplicities, is a linear combination of rational quadratic divisors. It defines an element of the Picard group of X_Γ. Let s be a generic point on a 1-dimensional cusp of X_Γ (cf. Definition 5.2.1). By pullback, the divisor of a Borcherds product also defines an element of the local Picard group $\mathrm{Pic}(X_\Gamma, s)$. If a divisor is torsion in all local Picard groups, we say that it is trivial at generic boundary points. Using our result regarding the basis problem for modular forms for the Weil representation, we can show that the space of local obstructions for constructing Borcherds products generates the space of global obstructions. This gives a condition for whether a given divisor is the divisor of a Borcherds product that depends only on its behaviour on the boundary components. Let $\Gamma = \ker(\mathrm{O}(L) \to \mathrm{O}(L'/L)) \cap \mathrm{O}(L)^+$ be the discriminant kernel of L. Generalizing [13, Theorem 5.4], we obtain Theorem 5.3.5:

Let L be an even lattice of signature $(n, 2)$ with even $n > 8$ splitting two hyperbolic planes $II_{1,1} \oplus II_{1,1}$. Assume that the discriminant form $D = L'/L$ satisfies the conditions of Theorem 4.3.7 for $m = n - 2$. Let

$$H = \frac{1}{2} \sum_{\beta \in D} \sum_{\substack{l \in \mathbb{Z}+\mathrm{q}(\beta) \\ l>0}} c(\beta, l) H(\beta, l)$$

be a finite linear combination of Heegner divisors $H(\beta, l)$ (with coefficients $c(\beta, l) \in \mathbb{Z}$). Then the following statements are equivalent:

i) H is the divisor of a Borcherds product for the group Γ.

ii) H is the divisor of a meromorphic automorphic form for Γ.

iii) H is trivial at generic boundary points.

As a corollary we get for certain lattices a new proof that ii) implies i), which is the converse theorem in [11]. Theorem 5.3.5 follows immediately from Theorem 4.3.7: Because of Serre duality, the existence of a Borcherds product with a given divisor is controlled by the space $\mathrm{S}_{1+n/2}(L'/L)$ and according to a result of Bruinier and Freitag in [13], whether a divisor is trivial at generic boundary points is controlled by $\Theta_{n-2,1+n/2}(L'/L)$.

This section is also based on the preprint [51].

Orthogonal Hecke operators

Let G be a group and $\Gamma \subset G$ be a subgroup. Then (Γ, G) is called a Hecke pair if for all $\alpha \in G$ the set

$$\Gamma \backslash \Gamma \alpha \Gamma$$

is finite. The Hecke algebra $\mathscr{H}(\Gamma, G)$ is then defined as the algebra of functions from $\Gamma \backslash G / \Gamma$ to $\mathbb{Z}$ with finite support and the product is given by a convolution product.

For example $(\mathrm{SL}_2(\mathbb{Z}), \mathrm{GL}_2(\mathbb{Q})_+)$ is a Hecke pair and the corresponding Hecke algebra acts on the space of modular forms for $\mathrm{SL}_2(\mathbb{Z})$. The theory of Hecke operators for symplectic groups has been studied intensively. For orthogonal modular forms much less is known. In [33] in particular the lattice $II_{10,2}$ was treated and in [43] some more general results were proved.

We will consider unimodular lattices, so let L be a unimodular even lattice of signature $(n,2)$ with $n \geq 3$ and let

$$\mathrm{GO}(L \otimes_{\mathbb{Z}} \mathbb{Q}) = \{\alpha \in \mathrm{GL}(L \otimes_{\mathbb{Z}} \mathbb{Q}) \mid (\alpha v, \alpha w) = s(\alpha)(v,w) \text{ for some } s(\alpha) \in \mathbb{Q}^\times\}$$

denote the group of orthogonal similitudes. We let $G = \mathrm{GO}(L \otimes_{\mathbb{Z}} \mathbb{Q})^+ \subset \mathrm{GO}(L \otimes_{\mathbb{Z}} \mathbb{Q})$ denote the subgroup consisting of elements with positive spinor norm and set $\Gamma = \mathrm{O}(L)^+$. Then (Γ, G) is a Hecke pair and $\mathscr{H}(\Gamma, G)$ acts on the space of orthogonal modular forms. First we prove an elementary divisor theorem for (Γ, G). Since scalar matrices act trivially on orthogonal modular forms, we may restrict to considering only those $\alpha \in G$ with $\alpha(L) \subset L$ and $s(\alpha) \in \mathbb{Z}_{>0}$. For $m \in \mathbb{Z}_{>0}$ we denote $G(m) = \{\alpha \in G \mid \alpha(L) \subset L,\ s(\alpha) = m\}$ and get a bijection

$$\begin{aligned} G(m)/\Gamma &\to \{M \subset L \mid M \cong \sqrt{m}L\} \\ \alpha &\mapsto \alpha(L). \end{aligned}$$

We then have Theorem 6.1.12:

Let $M \subset L$ with $M \cong \sqrt{m}L$ for some $m \in \mathbb{Z}_{>0}$. Then there exist unique integers $a_1 \mid \ldots \mid a_{n/2+1} \mid m$ with $a_i^2 \mid m$ for $i < n/2+1$ and $a_{n/2+1} \leq \sqrt{m}$ such that

$$M \in \Gamma L^{a_1,\ldots,a_{n/2+1};m}.$$

Here $L^{a_1,\ldots,a_{n/2+1};m} \cong \sqrt{m}L$ is a specific sublattice of L that depends on the integers $a_1 \mid \ldots \mid a_{n/2+1} \mid m$ (see page 162). We have a similar result for $\mathrm{SO}(L)^+$ (see Theorem 6.1.10).

By Theorem 6.1.12, if p is a prime, then $G(p)$ consists of only one double coset. We denote the corresponding Hecke operator by $\mathcal{T}(p)$. We will investigate them in more detail. In Proposition 6.2.3 we will give an explicit decomposition of the double coset $G(p)$ into right cosets modulo Γ. Using this decomposition we can show how $\mathcal{T}(p)$ commutes with the Φ-operator. We obtain Theorem 6.2.6:

Let χ be either trivial or equal to det*. Then the diagram*

$$\begin{array}{ccc} \mathcal{M}_k(\Gamma,\chi) & \xrightarrow{\sum_{j=1}^h r_j^\chi(p)\Phi_{w_j}} & \mathrm{M}_k \\ \downarrow{\scriptstyle \mathcal{T}(p)} & & \downarrow{\scriptstyle p^{n/2-k/2}T(p)+p^{n-k/2-1}+p^{k/2}} \\ \mathcal{M}_k(\Gamma,\chi) & \xrightarrow{\Phi_w} & \mathrm{M}_k \end{array}$$

commutes.

Here $T(p)$ is the classical Hecke operator on modular forms for $\mathrm{SL}_2(\mathbb{Z})$. The sum in the top arrow ranges over the Φ-operators for all 1-dimensional cusps of X_Γ and the r_j^χ are representation numbers counting numbers of certain sublattices.

Eigenvalues of Borcherds' Φ_{12}

An important example of an orthogonal modular form is Borcherds' Φ_{12}. Let $L = II_{26,2} = \Lambda \oplus II_{1,1} \oplus II_{1,1}$, where Λ is the Leech lattice. Then Φ_{12} is the multiplicative Borcherds lift of $1/\Delta$ on L, where $\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n$ is the modular discriminant. It is a holomorphic orthogonal modular form of weight 12 and character det for $\mathrm{O}(L)^+$ and it is the only holomorphic automorphic product of singular weight on a unimodular lattice. It is a simultaneous eigenform for $\mathscr{H}(\Gamma, G)$ with Γ and G as before. Using Theorem 6.2.6 we find that for the eigenvalues $\lambda(p)$ of Φ_{12} corresponding to $\mathcal{T}(p)$ we have

$$\lambda(p) = r(p)(p^7\tau(p) + p^{19} + p^6),$$

where $r(p)$ is equal to

$$\begin{aligned}\#\{M \subset \Lambda \mid \text{there exists a } \beta : \sqrt{p}\Lambda \xrightarrow{\sim} M \text{ such that } \det(\beta) = +p\} \\ - \#\{M \subset \Lambda \mid \text{there exists a } \beta : \sqrt{p}\Lambda \xrightarrow{\sim} M \text{ such that } \det(\beta) = -p\}.\end{aligned}$$

We will give an explicit formula for the representation numbers $r(p)$ in Theorem 7.3.2:

Let p be a prime and let a be a zero of $X^2 - \tau(p)p^{-11/2}X + 1$. Then the representation number $r(p)$ is up to a sign equal to

$$r(p) = \pm\frac{p^{33}}{a^6}\prod_{j=0}^{11}(1 + p^{-11/2+j}a).$$

We derive this formula with the help of the Satake isomorphism. The numbers $r(p)$ are up to a known factor equal to the Hecke eigenvalues of a Siegel theta function. The Satake parameters of this function, which encode its eigenvalues, were computed in [19].

The last two chapters are based on joint work with M. Dittmann and N. Scheithauer [22].

Part I

Modular forms for the Weil representation

Chapter 1

The Weil representation

In this chapter we describe the Weil representation and define modular forms that transform with respect to this representation. We will consider elliptic modular forms as well as Siegel modular forms. Finally, we also describe the adelic Weil representation.

1.1 Discriminant forms

In this section we recall some results on discriminant forms (cf. [1], [9], [20], [55], [60] and [66]).

A *discriminant form* is a finite abelian group D with a quadratic form $\mathrm{q} : D \to \mathbb{Q}/\mathbb{Z}$ such that $(\beta, \gamma) = \mathrm{q}(\beta + \gamma) - \mathrm{q}(\beta) - \mathrm{q}(\gamma) \bmod 1$ is a non-degenerate symmetric bilinear form. The *level* of D is the smallest positive integer N such that $N\,\mathrm{q}(\gamma) = 0 \bmod 1$ for all $\gamma \in D$. The *square class* of D is square if $|D|$ is a square and non-square otherwise.

Let L be an even lattice, i.e. a free $\mathbb{Z}$-module of finite rank with a bilinear form $(\cdot,\cdot) : L \times L \to \mathbb{Z}$ such that (λ, λ) is even for all $\lambda \in L$. We denote by $L' = \{\gamma \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid (\gamma, \lambda) \in \mathbb{Z} \text{ for all } \lambda \in L\}$ the *dual lattice* of L. Then L'/L is a discriminant form with the quadratic form given by $\mathrm{q}(\gamma) = (\gamma, \gamma)/2 \bmod 1$. Conversely, every discriminant form can be obtained in this way. The corresponding lattice can be chosen to be positive-definite. In general, $L \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{t_+} \times \mathbb{R}^{t_-}$ such that $(\cdot,\cdot)$ is positive-definite on $\mathbb{R}^{t_+}$ and negative-definite on $\mathbb{R}^{t_-}$ and (t_+, t_-) is called the *signature* of L. The signature $\mathrm{sign}(D) \in \mathbb{Z}/8\mathbb{Z}$ of a discriminant form D is defined as $\mathrm{sign}(D) = t_+ - t_- \bmod 8$, where (t_+, t_-) is the signature of any even lattice with that discriminant form.

Every discriminant form decomposes into a sum of Jordan components and every

Jordan component can be written as a sum of indecomposable Jordan components (usually not uniquely). The possible non-trivial Jordan components are the following.

Let $q > 1$ be a power of an odd prime p. The non-trivial p-adic Jordan components of exponent q are $q^{\pm n}$ for $n \geq 1$. The indecomposable components are $q^{\pm 1}$, generated by an element γ with $q\gamma = 0$, $\mathrm{q}(\gamma) = a/q \mod 1$ where a is an integer with $\left(\frac{2a}{p}\right) = \pm 1$. These components all have level q. The p-excess is given by $p\text{-excess}(q^{\pm n}) = n(q-1) + 4k \mod 8$ where $k = 1$ if q is not a square and the exponent is $-n$, and $k = 0$ otherwise. We define $\gamma_p(q^{\pm n}) = e(-p\text{-excess}(q^{\pm n})/8)$.

Let $q > 1$ be a power of 2. The non-trivial even 2-adic Jordan components of exponent q are $q^{\pm 2n} = q_{II}^{\pm 2n}$ for $n \geq 1$. The indecomposable components are $q_{II}^{\pm 2}$ generated by two elements γ and δ with $q\gamma = q\delta = 0$, $(\gamma, \delta) = 1/q \mod 1$ and $\mathrm{q}(\gamma) = \mathrm{q}(\delta) = 0 \mod 1$ for q_{II}^{+2} and $\mathrm{q}(\gamma) = \mathrm{q}(\delta) = 1/q \mod 1$ for q_{II}^{-2}. These components all have level q. The oddity is given by $\mathrm{oddity}(q_{II}^{\pm 2n}) = 4k \mod 8$ with $k = 1$ if q is not a square and the exponent is $-2n$, and $k = 0$ otherwise. We define $\gamma_2(q_{II}^{\pm 2n}) = e(\mathrm{oddity}(q_{II}^{\pm 2n})/8)$.

Let $q > 1$ be a power of 2. The non-trivial odd 2-adic Jordan components of exponent q are $q_t^{\pm n}$ with $n \geq 1$ and $t \in \mathbb{Z}/8\mathbb{Z}$. If $n = 1$, then $\pm = +$ implies $t = \pm 1 \mod 8$ and $\pm = -$ implies $t = \pm 3 \mod 8$. If $n = 2$, then $\pm = +$ implies $t = 0$ or $\pm 2 \mod 8$ and $\pm = -$ implies $t = 4$ or $\pm 2 \mod 8$. For any n we have $t = n \mod 2$. The indecomposable components are $q_t^{\pm 1}$ where $\left(\frac{t}{2}\right) = \pm 1$ (recall that $\left(\frac{t}{2}\right) = +1$ if $t = \pm 1 \bmod 8$ and $\left(\frac{t}{2}\right) = -1$ if $t = \pm 3 \bmod 8$) generated by an element γ with $q\gamma = 0$, $\mathrm{q}(\gamma) = t/2q \mod 1$. These components all have level $2q$. The oddity is given by $\mathrm{oddity}(q_t^{\pm n}) = t + 4k \mod 8$ with $k = 1$ if q is not a square and the exponent is $-n$, and $k = 0$ otherwise. We define $\gamma_2(q_t^{\pm n}) = e(\mathrm{oddity}(q_t^{\pm n})/8)$.

The sum of two Jordan components with the same prime power q is given by multiplying the signs, adding the ranks and if any components have a subscript t, adding the subscripts t. Isomorphic discriminant forms can have different 2-adic symbols.

Let D be a discriminant form. Then

$$\mathrm{sign}(D) + \sum_{p \geq 3} p\text{-excess}(D) = \mathrm{oddity}(D) \mod 8,$$

respectively

$$\prod \gamma_p(D) = e(\mathrm{sign}(D)/8)\,.$$

We will also use

$$e(\mathrm{oddity}(D)/4) = \left(\frac{-1}{|D|}\right) e(\mathrm{sign}(D)/4)\,.$$

Let c be an integer. Then c acts by multiplication on D and we have an exact sequence $0 \to D_c \to D \to D^c \to 0$ where D_c is the kernel and D^c the image of this map. Note that D^c is the orthogonal complement of D_c.

The set $D^{c*} = \{\gamma \in D \,|\, c\,\mathrm{q}(\alpha) + (\alpha,\gamma) = 0 \text{ for all } \alpha \in D_c\}$ is a coset of D^c. Let $2^k \| c$. After a choice of Jordan decomposition we set $x_c = 0$ if the Jordan block of type 2^k is even and $x_c = (2^{k-1}, \dots, 2^{k-1})$ in this block if it is odd. Then x_c is a canonical coset representative of D^{c*}. We can write $\gamma \in D^{c*}$ as $\gamma = x_c + c\mu$ and $\mathrm{q}_c(\gamma) = c\,\mathrm{q}(\mu) + x_c\mu \mod 1$ is well-defined (see [60]). If c is even, then $\mathcal{D}_{c/2} \subset \mathcal{D}_c$ and for $\alpha \in \mathcal{D}_{c/2}$ we have $c\,\mathrm{q}(\alpha) = 0 \bmod 1$, so that $\mathcal{D}^{c*} \subset \{\gamma \in \mathcal{D} \mid (\alpha,\gamma) = 0 \bmod 1 \text{ for all } \alpha \in \mathcal{D}_{c/2}\} = \mathcal{D}^{c/2}$.

For two discriminant forms D, D' with quadratic forms q and q$'$ such that $D \cong D'$ we define

$$\begin{aligned}\mathrm{Iso}(D,D') \coloneqq \{\sigma : D \to D' \mid \sigma &\text{ is a group isomorphism with} \\ &\mathrm{q}'(\sigma(\gamma)) = \mathrm{q}(\gamma) \bmod 1 \text{ for all } \gamma \in D\}\end{aligned}$$

and $\mathrm{O}(D) \coloneqq \mathrm{Iso}(D,D)$.

We describe the number of elements of a given norm in p-elementary discriminant forms. To simplify the notation we define for $x \in \mathbb{Q}/\mathbb{Z}$

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \mod 1, \\ 0 & \text{if } x \neq 0 \mod 1. \end{cases}$$

For odd primes we have (see Proposition 3.2 in [59])

Proposition 1.1.1. *Let p be an odd prime. Then the number $N(p^{\epsilon n}, j)$ of elements of norm $j/p \mod 1$ in the discriminant form $p^{\epsilon n}$ is given by*

$$N(p^{\epsilon n}, j) = \begin{cases} p^{n-1} + \epsilon\left(\frac{-1}{p}\right)^{n/2} \big(p\delta(j/p) - 1\big)p^{(n-2)/2} & \text{if } n \text{ is even,} \\ p^{n-1} + \epsilon\left(\frac{-1}{p}\right)^{(n-1)/2} \left(\frac{2}{p}\right)\left(\frac{j}{p}\right) p^{(n-1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

In the level 2 case we have (see Proposition 3.1 in [59])

Proposition 1.1.2. *The number of elements of norm $j/2 \mod 1$ in $2_{II}^{\epsilon n}$ is given by*

$$N(2_{II}^{\epsilon n}, j) = 2^{n-1} + \epsilon(-1)^j 2^{(n-2)/2}\,.$$

Finally, for the level 4 case

Proposition 1.1.3. *The number of elements of norm* $j/4 \mod 1$ *in* $2_t^{\epsilon n}$ *is given by*

$$N(2_t^{\epsilon n}, j) = \begin{cases} 2^{n-2} + \epsilon \left(\frac{t}{2}\right) 2^{(n-3)/2} & \text{if } j = 0 \mod 4, \\ 2^{n-2} - \epsilon \left(\frac{t}{2}\right) 2^{(n-3)/2} & \text{if } j = 2 \mod 4, \\ 2^{n-2} + \epsilon \left(\frac{t}{2}\right) (-1)^{(t-1)/2} 2^{(n-3)/2} & \text{if } j = 1 \mod 4, \\ 2^{n-2} - \epsilon \left(\frac{t}{2}\right) (-1)^{(t-1)/2} 2^{(n-3)/2} & \text{if } j = 3 \mod 4, \end{cases}$$

if n is odd and by

$$N(2_t^{\epsilon n}, j) = \begin{cases} 2^{n-2} + \epsilon\delta(t/4) \left(\frac{t-1}{2}\right) 2^{(n-2)/2} & \text{if } j = 0 \mod 4, \\ 2^{n-2} - \epsilon\delta(t/4) \left(\frac{t-1}{2}\right) 2^{(n-2)/2} & \text{if } j = 2 \mod 4, \\ 2^{n-2} + \epsilon\delta((t+2)/4) \left(\frac{t-1}{2}\right) 2^{(n-2)/2} & \text{if } j = 1 \mod 4, \\ 2^{n-2} - \epsilon\delta((t+2)/4) \left(\frac{t-1}{2}\right) 2^{(n-2)/2} & \text{if } j = 3 \mod 4, \end{cases}$$

if n is even.

Proof: As in the previous cases this can be proved by induction on n. □

1.2 The Weil representation of $\mathrm{Mp}_2(\mathbb{Z})$

The metaplectic group $\mathrm{Mp}_2(\mathbb{R})$ is the unique connected double cover of the group $\mathrm{SL}_2(\mathbb{R})$. Its elements can be written as pairs (M, ϕ) where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and ϕ is a holomorphic function on the upper half-plane $\mathbb{H}$ such that $\phi(\tau)^2 = j(M, \tau) := c\tau + d$. Then the product of two elements in $\mathrm{Mp}_2(\mathbb{R})$ is given by

$$(M_1, \phi_1(\tau))(M_2, \phi_2(\tau)) = (M_1 M_2, \phi_1(M_2\tau)\phi_2(\tau)),$$

where $M\tau = \frac{a\tau+b}{c\tau+d}$ denotes the *Möbius transform* on $\mathbb{H}$. The inverse image of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{Mp}_2(\mathbb{R})$ is denoted by $\mathrm{Mp}_2(\mathbb{Z})$. The standard generators of $\mathrm{Mp}_2(\mathbb{Z})$ are $S = (\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau})$ and $T = (\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1)$, where $\sqrt{\cdot}$ denotes the principal branch. Let D be a discriminant form of level N and let $\mathbb{C}[D]$ be its group ring generated by a formal basis $(e^\gamma)_{\gamma \in D}$. Then the *Weil representation* of $\mathrm{Mp}_2(\mathbb{Z})$ is defined as (cf. [7])

$$\rho_D(T)e^\gamma = e(\mathrm{q}(\gamma))\, e^\gamma$$

$$\rho_D(S)e^\gamma = \frac{e(-\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta \in D} e(-(\gamma, \beta))\, e^\beta,$$

where $e(z) := e^{2\pi iz}$. For discriminant forms of even signature the inverse image of the group $\Gamma(N) = \{M \in \mathrm{SL}_2(\mathbb{Z}) \mid M = I \bmod N\}$ under the covering map $\mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$ acts trivially in the Weil representation. Again, let $\sqrt{\cdot}$ denote the principal branch. Then

$$s : \Gamma_1(4) \to \mathrm{Mp}_2(\mathbb{Z}), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \left(\frac{c}{d}\right) \sqrt{c\tau + d} \right)$$

is a section of $\Gamma_1(4)$ under the covering map. For odd signature we must have $4 \mid N$ and s defines the unique section of $\Gamma(N)$ such that $s(\Gamma(N))$ acts trivially in the Weil representation. We denote the corresponding group in both cases by $\mathrm{Mp}_2(N)$. The quotient $\mathrm{Mp}_2(\mathbb{Z})/\mathrm{Mp}_2(N)$ is isomorphic to $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ for even signature and to a double cover of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ for odd signature.

We define a scalar product on the group ring $\mathbb{C}[D]$ which is linear in the first and antilinear in the second variable by

$$\langle e^\gamma, e^\beta \rangle = \begin{cases} 1 & \text{if } \gamma = \beta \\ 0 & \text{otherwise.} \end{cases}$$

The Weil representation is unitary with respect to this scalar product.

A $\sigma \in \mathrm{Iso}(D, D')$ induces a *pullback* $\sigma^* : \mathbb{C}[D'] \to \mathbb{C}[D]$ and a *pushforward* $\sigma_* : \mathbb{C}[D] \to \mathbb{C}[D']$ by

$$\sigma^* e^\gamma = e^{\sigma^{-1}\gamma} \text{ and}$$
$$\sigma_* e^\gamma = e^{\sigma\gamma}$$

respectively. The pushforward defines a unitary representation of $\mathrm{O}(D)$ on $\mathbb{C}[D]$ that commutes with the Weil representation.

The kernel of the covering map $\mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$ is given by $\{(I, \pm 1)\}$ and the element $(I, -1)$ acts in the Weil representation as multiplication by $e(\mathrm{sign}(D)/2)$. Let us for now assume that the signature of the discriminant form is even. Then $\{(I, \pm 1)\}$ acts trivially so that the Weil representation descends to a representation of $\mathrm{SL}_2(\mathbb{Z})$ and we identify S, respectively T with their projection to $\mathrm{SL}_2(\mathbb{Z})$.

The element $Z = S^2 = -I$ acts as

$$\rho_D(Z) e^\gamma = e(\mathrm{sign}(D)/4) e^{-\gamma} .$$

For a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$\rho_D(M) e^\gamma = \xi(M) \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\beta \in D^{c*}} e(a\,\mathrm{q}_c(\beta))\, e(b(\beta, \gamma))\, e(bd\,\mathrm{q}(\gamma))\, e^{d\gamma + \beta} \tag{1.2.1}$$

with $\xi(M) = e(\mathrm{sign}(D)/4)\prod \xi_p$. The local factors ξ_p can be expressed in terms of the Jordan components of D (see [60, Theorem 4.7], note however that in [60] the dual Weil representation was used).

Let N be a positive integer such that the level of D divides N. If $c = 0 \mod N$, the above formula simplifies to

$$\rho_D(M)e^\gamma = \chi_D(a)\, e(bd\, \mathrm{q}(\gamma))\, e^{d\gamma}$$

where

$$\chi_D(a) = \left(\frac{a}{|D|}\right) e((a-1)\,\mathrm{oddity}(D)/8\,)$$

is a quadratic Dirichlet character modulo N. In particular one sees that $\Gamma(N)$ acts trivially.

The formula

$$\sum_{\gamma \in D} e(\mathrm{q}(\gamma)) = e(\mathrm{sign}(D)/8)\sqrt{|D|}$$

is known as Milgram's formula. For $c \in \mathbb{Z}$ with $(c, N) = 1$ it follows that

$$\sum_{\gamma \in D} e(c\, \mathrm{q}(\gamma)) = \chi_D(c) e(\mathrm{sign}(D)/8)\sqrt{|D|}.$$

A formula for general Gauss sums can be found in [60, Theorem 3.9].

1.3 Elliptic modular forms for the Weil representation

We now want to define modular forms for the Weil representation. Let

$$\mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$$

be the complex upper half-plane. Let $k \in \frac{1}{2}\mathbb{Z}$ and f be a function from $\mathbb{H}$ to a complex vector space. For $(M, \phi) \in \mathrm{Mp}_2(\mathbb{R})$ we define the Petersson-slash operator $|_k$ by

$$f|_k[(M, \phi)](\tau) := \phi(\tau)^{-2k} f(M\tau).$$

Definition 1.3.1. Let D be a discriminant form and let $k \in \frac{1}{2}\mathbb{Z}$. A function $f : \mathbb{H} \to \mathbb{C}[D]$ is called a *modular form of weight k with respect to ρ_D and* $\mathrm{Mp}_2(\mathbb{Z})$ if

(i) $f|_k[(M, \phi)] = \rho_D((M, \phi))f$ for all $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$,

(ii) f is holomorphic on $\mathbb{H}$,

(iii) f is holomorphic at the cusp ∞.

Here condition (iii) means that f has a Fourier expansion of the form

$$f(\tau) = \sum_{\gamma \in D} \sum_{\substack{n \in \mathbb{Z}+\mathrm{q}(\gamma) \\ n \geq 0}}^{\infty} c(\gamma, n) e(n\tau) e^{\gamma}.$$

Moreover, if all $c(\gamma, n)$ with $n = 0$ vanish, then f is called a cusp form. The $\mathbb{C}$-vector space of modular forms of weight k with respect to ρ_D and $\mathrm{Mp}_2(\mathbb{Z})$ is denoted by $\mathrm{M}_k(D)$, the subspace of cusp forms by $\mathrm{S}_k(D)$.

Note that there are non-zero modular forms of weight k for ρ_D only if $2k = \mathrm{sign}(D) \bmod 2$. If k is an integer and $\mathrm{sign}(D)$ is even, then the kernel of the covering map $\mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$ acts trivially in both the Weil representation and the slash operator, so that we can consider modular forms for the group $\mathrm{SL}_2(\mathbb{Z})$.

For $f, g \in \mathrm{M}_k(D)$, where at least one of f and g is in $\mathrm{S}_k(D)$ we define the *Petersson inner product* $(\cdot, \cdot)$ by

$$(f, g) = \int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} \langle f(\tau), g(\tau) \rangle y^k \frac{\mathrm{d}x \mathrm{d}y}{y^2},$$

where $\tau = x + iy$. On $\mathrm{S}_k(D)$ the Petersson inner product defines a scalar product.

We furthermore denote by $\mathrm{M}_k(\mathrm{Mp}_2(N))$ the space of scalar-valued modular forms for the subgroup $\mathrm{Mp}_2(N)$. Recall that $\mathrm{Mp}_2(N)$ depends on D. Let $f \in \mathrm{M}_k(\mathrm{Mp}_2(N))$ and $v \in \mathbb{C}[D]$. Then

$$F_{f,v} := \frac{1}{|\mathrm{Mp}_2(N) \backslash \mathrm{Mp}_2(\mathbb{Z})|} \sum_{\substack{(M,\phi) \in \\ \mathrm{Mp}_2(N) \backslash \mathrm{Mp}_2(\mathbb{Z})}} f|_k(M, \phi) \rho_D(M, \phi)^{-1} v \in \mathrm{M}_k(D)$$

is a modular form for the Weil representation. These modular forms span $\mathrm{M}_k(D)$, when f ranges over $\mathrm{M}_k(\mathrm{Mp}_2(N))$ and v ranges over $\mathbb{C}[D]$ (This is a classical idea, cf. e.g. [64], in the case of the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ see [61]).

It is often convenient to write $f \in \mathrm{M}_k(D)$ in terms of its component functions $f = \sum_{\gamma \in D} f_\gamma e^\gamma$. Then $f_\gamma = \langle f, e^\gamma \rangle \in \mathrm{M}_k(\mathrm{Mp}_2(N))$. For a proof confer the proof of Proposition 2.2.1. Note that in general $\langle F_{f,e^\gamma}, e^\gamma \rangle \neq f$.

We can also define vector-valued *Eisenstein series* and *Poincaré series*. Let $k > 2$ and let $\gamma \in D$ be isotropic. Then the Eisenstein series

$$E_{k,D,\gamma}(\tau) := \frac{1}{2} \sum_{(M,\phi) \in \Gamma_\infty^+ \backslash \mathrm{Mp}_2(\mathbb{Z})} e^\gamma|_k[(M, \phi)](\tau)$$

with $\Gamma_\infty^+ = \langle T \rangle$ converges normally on $\mathbb{H}$ and therefore defines a modular form of weight k for ρ_D. For any $\gamma \in D$ let $m \in \mathbb{Z} + \mathrm{q}(\gamma)$ with $m > 0$. Then also the Poincaré series

$$P_{k,D,\gamma,m}(\tau) := \frac{1}{2} \sum_{(M,\phi)\in\Gamma_\infty^+\backslash \mathrm{Mp}_2(\mathbb{Z})} e(m\cdot)e^\gamma|_k[(M,\phi)](\tau)$$

converges normally on $\mathbb{H}$ and defines a cusp form of weight k. In [11] it was shown that for any cusp form $f \in \mathrm{S}_k(D)$ with Fourier coefficients $c(\gamma, m)$ one has

$$(f, P_{k,D,\gamma,m}) = 2\frac{(k-2)!}{(4\pi m)^{k-1}} c(\gamma, m).$$

1.4 The Weil representation of $\mathrm{Mp}_{2n}(\mathbb{Z})$

The Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ is a special case of a more general class of representations for $\mathrm{Sp}_{2n}(\mathbb{Z})$. For discriminant forms of odd signature, again, one needs to consider the double cover $\mathrm{Mp}_{2n}(\mathbb{Z})$ of $\mathrm{Sp}_{2n}(\mathbb{Z})$. For simplicity, we will only consider even signature.

Now we want to recall some facts about the symplectic group. A nice reference is [30].

Let $n \in \mathbb{Z}_{>0}$ and let $J = J_n = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$, where $I = I_n$ is the identity matrix of rank n. The symplectic group $\mathrm{Sp}_{2n}(\mathbb{Z})$ is defined as

$$\Gamma^{(n)} := \mathrm{Sp}_{2n}(\mathbb{Z}) := \{M \in \mathrm{GL}_{2n}(\mathbb{Z}) \mid M^T J M = J\}.$$

A matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with $A, B, C, D \in \mathrm{Mat}_n(\mathbb{Z})$ is in $\mathrm{Sp}_{2n}(\mathbb{Z})$ if and only if

$$A^T D - C^T B = D^T A - B^T C = I, \quad A^T C = C^T A, \quad B^T D = D^T B$$

or equivalently

$$AD^T - BC^T = DA^T - CB^T = I, \quad AB^T = BA^T, \quad CD^T = DC^T,$$

in particular $\mathrm{Sp}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$. Furthermore, for $M \in \mathrm{Sp}_{2n}(\mathbb{Z})$ also $M^T \in \mathrm{Sp}_{2n}(\mathbb{Z})$ and $\det(M) = 1$. We have $J^{-1} = J^T = -J$ and in general the inverse of a symplectic matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ (notation as above) is given by

$$M^{-1} = J^{-1} M^T J = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}.$$

We define maps

$$n : \mathrm{Sym}_n(\mathbb{Z}) \to \Gamma^{(n)}, \quad n(S) = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}$$

$$a : \mathrm{GL}_n(\mathbb{Z}) \to \Gamma^{(n)}, \quad a(U) = \begin{pmatrix} U & 0 \\ 0 & (U^T)^{-1} \end{pmatrix}$$

$$u : \Gamma^{(n-1)} \to \Gamma^{(n)}, \quad u\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$d : \Gamma^{(n-1)} \to \Gamma^{(n)}, \quad d\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 1 & 0 \\ 0 & C & 0 & D \end{pmatrix}.$$

Then n, a, u and d are group homomorphisms and $u(M)d(M') = d(M')u(M)$. The symplectic group $\Gamma^{(n)}$ is generated by J_n and matrices of the form $n(S)$ for $S \in \mathrm{Sym}_n(\mathbb{Z})$. The subgroup

$$\Gamma^{(n)}_\infty := \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma^{(n)} \right\}$$

is generated by elements of the form $n(S)$ and $a(U)$. In the case $n = 2$ we will later need the matrix

$$\mathcal{A}_l := J_2 \cdot n\left(\begin{pmatrix} 0 & -1 \\ -1 & -l \end{pmatrix}\right) J_2 \cdot n\left(\begin{pmatrix} l^2+l & -l-1 \\ -l-1 & 1 \end{pmatrix}\right) J_2 \cdot n\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$$
$$= \begin{pmatrix} l^2+l & -l-1 & -1 & -l-1 \\ -l-1 & 1 & 0 & 0 \\ -l & 1 & 0 & 0 \\ 0 & 0 & -1 & -l \end{pmatrix}.$$

Note that the matrix $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ from the previous section is equal to $-J_1$ and should not be confused with some $S \in \mathrm{Sym}_n(\mathbb{Z})$. To be consistent, we will from now on always work with J instead of S. Furthermore, note that we use D both for discriminant forms and for the lower right entry of a symplectic matrix. Since the nature of these objects is entirely different, there should be no danger of confusing them.

Let D be a discriminant form of even signature and level N and let $\mathbb{C}[D^n]$ be the group algebra of $D^n = D \times \ldots \times D$ spanned by a formal basis $(e^{\underline{\gamma}})_{\underline{\gamma} \in D^n}$, where

$\underline{\gamma} = (\gamma_1, \dots, \gamma_n)$. Then the *Weil representation* $\rho_D^{(n)} : \mathrm{Sp}_{2n}(\mathbb{Z}) \to \mathbb{C}[D^n]$ of $\mathrm{Sp}_{2n}(\mathbb{Z})$ can be defined by (cf. [69], [7] and [74])

$$\begin{aligned} \rho_D^{(n)}(n(S))e^{\underline{\gamma}} &= e(1/2\,\mathrm{tr}(S(\underline{\gamma},\underline{\gamma})))e^{\underline{\gamma}} \\ \rho_D^{(n)}(J)e^{\underline{\gamma}} &= \frac{e(n\,\mathrm{sign}(D)/8)}{\sqrt{|D|^n}} \sum_{\underline{\beta}\in D^n} e(\mathrm{tr}(\underline{\gamma},\underline{\beta}))\, e^{\underline{\beta}}, \end{aligned}$$

where $(\underline{\gamma},\underline{\beta}) = ((\gamma_i,\beta_j))_{i,j=1}^n \in \mathrm{Mat}_n(\mathbb{Q}/\mathbb{Z})$. This implies

$$\rho_D^{(n)}(a(U))e^{\underline{\gamma}} = \det(U)^{\mathrm{sign}(D)/2} e^{\underline{\gamma}U^{-1}}$$

and in particular $\rho_D^{(n)}(-I)e^{\underline{\gamma}} = e(n\,\mathrm{sign}(D)/4)e^{-\underline{\gamma}}$.

We define a scalar product on the group algebra $\mathbb{C}[D^n]$ which is linear in the first and antilinear in the second variable by

$$\langle e^{\underline{\gamma}}, e^{\underline{\beta}}\rangle = \begin{cases} 1 & \text{if } \underline{\gamma} = \underline{\beta} \\ 0 & \text{otherwise.} \end{cases}$$

Then the Weil representation is unitary with respect to this scalar product. There is a natural isomorphism $\mathbb{C}[D^n] \cong \mathbb{C}[D]^{\otimes n}$ that we will make frequent use of. For the following cf. [67, Lemma 3.4].

Proposition 1.4.1. *For a symplectic matrix $M \in \Gamma^{(n-1)}$ the symplectic matrices $u(M)$ and $d(M)$ transform in the Weil representation as*

$$\begin{aligned} \rho_D^{(n)}(u(M))e^{(\gamma_1,\dots,\gamma_n)} &= \rho_D^{(n-1)}(M)e^{(\gamma_1,\dots,\gamma_{n-1})} \otimes e^{\gamma_n} \\ \rho_D^{(n)}(d(M))e^{(\gamma_1,\dots,\gamma_n)} &= e^{\gamma_1} \otimes \rho_D^{(n-1)}(M)e^{(\gamma_2,\dots,\gamma_n)}. \end{aligned}$$

Proof. It suffices to prove the assertion for the generators $n(S)$ and J_{n-1} of $\Gamma^{(n-1)}$. For $S \in \mathrm{Sym}_{n-1}(\mathbb{Z})$ we have

$$\begin{aligned} u(n(S)) &= n(S_1) \\ d(n(S)) &= n(S_2) \end{aligned}$$

with $S_1 = \left(\begin{smallmatrix} S & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $S_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & S \end{smallmatrix}\right)$ and the identity is trivial. Furthermore, we find

$$\begin{aligned} u(J_{n-1}) &= -n\left(\begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix}\right) J_n n\left(\begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix}\right) J_n n(I_n) \\ d(J_{n-1}) &= -n\left(\begin{pmatrix} -1 & 0 \\ 0 & I_{n-1} \end{pmatrix}\right) J_n n\left(\begin{pmatrix} 0 & 0 \\ 0 & I_{n-1} \end{pmatrix}\right) J_n n(I_n) \end{aligned}$$

and compute

$$
\begin{aligned}
&\rho_D^{(n)}(u(J_{n-1}))e^{(\gamma_1,\ldots,\gamma_n)}\\
&\quad= \rho_D^{(n)}\left(-n\left(\left(\begin{smallmatrix} I_{n-1} & 0 \\ 0 & -1\end{smallmatrix}\right)\right) J_n n\left(\left(\begin{smallmatrix} I_{n-1} & 0 \\ 0 & 0\end{smallmatrix}\right)\right) J_n\right) e\left(\sum_{i=1}^{n} \mathrm{q}(\gamma_i)\right) e^{(\gamma_1,\ldots,\gamma_n)}\\
&\quad= \rho_D^{(n)}\left(-n\left(\left(\begin{smallmatrix} I_{n-1} & 0 \\ 0 & -1\end{smallmatrix}\right)\right) J_n n\left(\left(\begin{smallmatrix} I_{n-1} & 0 \\ 0 & 0\end{smallmatrix}\right)\right)\right) \frac{e(n\,\mathrm{sign}(D)/8)}{\sqrt{|D|^n}}\\
&\qquad\qquad \sum_{(\beta_1,\ldots,\beta_n)\in D^n} e\left(\sum_{i=1}^{n}(\mathrm{q}(\gamma_i)+(\gamma_i,\beta_i))\right) e^{(\beta_1,\ldots,\beta_n)}\\
&\quad= \rho_D^{(n)}\left(-n\left(\left(\begin{smallmatrix} I_{n-1} & 0 \\ 0 & -1\end{smallmatrix}\right)\right) J_n\right) \frac{e(n\,\mathrm{sign}(D)/8)}{\sqrt{|D|^n}}\\
&\qquad\qquad \sum_{(\beta_1,\ldots,\beta_n)\in D^n} e\left(\sum_{i=1}^{n-1}\mathrm{q}(\gamma_i+\beta_i)\right) e(\mathrm{q}(\gamma_n)+(\gamma_n,\beta_n))e^{(\beta_1,\ldots,\beta_n)}\\
&\quad= \rho_D^{(n)}\left(-n\left(\left(\begin{smallmatrix} I_{n-1} & 0 \\ 0 & -1\end{smallmatrix}\right)\right)\right) \frac{e(n\,\mathrm{sign}(D)/4)}{|D|^n} \sum_{\substack{(\beta_1,\ldots,\beta_n)\in D^n\\ (\mu_1,\ldots,\mu_n)\in D^n}}\\
&\qquad\qquad e\left(\sum_{i=1}^{n}(\beta_i,\mu_i)\right) e\left(\sum_{i=1}^{n-1}\mathrm{q}(\gamma_i+\beta_i)\right) e(\mathrm{q}(\gamma_n)+(\gamma_n,\beta_n))e^{(\mu_1,\ldots,\mu_n)}\\
&\quad= \rho_D^{(n)}(-I)\frac{e(n\,\mathrm{sign}(D)/4)}{|D|^n} \sum_{\substack{(\beta_1,\ldots,\beta_n)\in D^n\\ (\mu_1,\ldots,\mu_n)\in D^n}} e\left(\sum_{i=1}^{n-1}(\mathrm{q}(\gamma_i+\beta_i+\mu_i)-(\gamma_i,\mu_i))\right)\\
&\qquad\qquad e(\mathrm{q}(\gamma_n)-\mathrm{q}(\mu_n))e((\gamma_n+\mu_n,\beta_n))e^{(\mu_1,\ldots,\mu_n)}.
\end{aligned}
$$

If β_i ranges over all elements in D, then so does $\gamma_i+\beta_i+\mu_i$. We can use Milgram's formula to evaluate the sum over $(\beta_1,\ldots,\beta_{n-1})\in D^{n-1}$ and get

$$
\begin{aligned}
\rho_D^{(n)}(-I)\frac{e((3n-1)\,\mathrm{sign}(D)/8)}{\sqrt{|D|^{n-1}}|D|} &\sum_{(\mu_1,\ldots,\mu_n)\in D^n} e\left(-\sum_{i=1}^{n-1}(\gamma_i,\mu_i)\right)\\
e(\mathrm{q}(\gamma_n)-\mathrm{q}(\mu_n))e^{(\mu_1,\ldots,\mu_n)} &\sum_{\beta_n\in D} e((\gamma_n+\mu_n,\beta_n)).
\end{aligned}
$$

Now the last sum is a character sum which is equal to $|D|$ if $\mu_n=-\gamma_n$ and 0 otherwise. Thus, we have

$$\rho_D^{(n)}(-I)\frac{e((3n-1)\operatorname{sign}(D)/8)}{\sqrt{|D|^{n-1}}} \sum_{(\mu_1,\dots,\mu_{n-1})\in D^{n-1}} e\left(-\sum_{i=1}^{n-1}(\gamma_i,\mu_i)\right) e^{(\mu_1,\dots,\mu_{n-1})}\otimes e^{-\gamma_n}$$
$$= \frac{e((5n-1)\operatorname{sign}(D)/8)}{\sqrt{|D|^{n-1}}} \sum_{(\mu_1,\dots,\mu_{n-1})\in D^{n-1}} e\left(\sum_{i=1}^{n-1}(\gamma_i,\mu_i)\right) e^{(\mu_1,\dots,\mu_{n-1})}\otimes e^{\gamma_n}$$
$$= \rho_D^{(n-1)}(J_{n-1})e^{(\gamma_1,\dots,\gamma_{n-1})}\otimes e^{\gamma_n},$$

where in the last step we used that $\operatorname{sign}(D)$ is even. A similar calculation shows the assertion for $d(J_{n-1})$. □

As for $n=1$, an element $\sigma \in \operatorname{Iso}(D,D')$ induces a *pullback* $\sigma^* : \mathbb{C}[D'^n] \to \mathbb{C}[D^n]$ and a *pushforward* $\sigma_* : \mathbb{C}[D^n] \to \mathbb{C}[D'^n]$ by

$$\sigma^* e^{\underline{\gamma}} = e^{\sigma^{-1}\underline{\gamma}} \text{ and}$$
$$\sigma_* e^{\underline{\gamma}} = e^{\sigma\underline{\gamma}}$$

respectively. The pushforward defines a unitary representation of $\mathrm{O}(D)$ on $\mathbb{C}[D^n]$ that commutes with the Weil representation.

We have seen that the Weil representation of $\mathrm{Sp}_{2n}(\mathbb{Z})$ generalizes the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$. For notation such as $\mathbb{H}_n$, $\rho_D^{(n)}$ or $\Gamma^{(n)}$, whenever n is omitted, it is assumed to be equal to 1.

1.5 Siegel modular forms for the Weil representation

We will now define modular forms for the Weil representation of $\mathrm{Sp}_{2n}(\mathbb{Z})$ generalizing the definition from [30] to the vector-valued case (cf. also [11] and [74]). Let

$$\mathbb{H}_n := \{Z \in \mathrm{Sym}_n(\mathbb{C}) \mid \mathrm{Im}(Z) \in \mathrm{Pos}_n(\mathbb{R})\}$$

be the Siegel half-space. Let $k \in \mathbb{Z}$ and f be a function from $\mathbb{H}$ to a complex vector space. Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GL}_{2n}(\mathbb{R})$ be a symplectic similitude matrix, i.e. $M^T J_n M = l J_n$ for some $l \in \mathbb{R}_{>0}$. We define the *factor of automorphy* $j(M,Z) := \det(CZ+D)$ and define the Petersson-slash operator $|_k$ by

$$(f|_k[M])(Z) = \det(M)^{k/2}\det(CZ+D)^{-k} f((AZ+B)(CZ+D)^{-1}).$$

The factor of automorphy satisfies the cocycle relation

$$j(M_1, M_2Z)j(M_2, Z) = j(M_1M_2, Z)$$

for symplectic similitude matrices $M_1, M_2 \in \mathrm{GL}_{2n}(\mathbb{R})$.

Definition 1.5.1. Let D be a discriminant form of even signature and level N and let $k \in \mathbb{Z}$. A function $f : \mathbb{H}_n \to \mathbb{C}[D^n]$ is called a *modular form of weight k with respect to $\rho_D^{(n)}$ and* $\mathrm{Sp}_{2n}(\mathbb{Z})$ if

(i) $f|_k[M] = \rho_D^{(n)}(M)f$ for all $M \in \mathrm{Sp}_{2n}(\mathbb{Z})$,

(ii) f is holomorphic on $\mathbb{H}_n$,

(iii) f is bounded on all domains of type $\mathrm{Im}(Z) \geq Y_0, Y_0 > 0$.

When $n > 1$ condition (iii) already follows from (i) and (ii) by the Koecher principle. Condition (iii) is equivalent to the fact that f has a Fourier expansion of the form

$$f(Z) = \sum_{\underline{\gamma} \in D^n} \sum_{\substack{S=S^T \text{ even} \\ S \geq 0}} c(\underline{\gamma}, S) e\left(\frac{\mathrm{tr}(SZ)}{2N}\right) e_{\underline{\gamma}}.$$

Moreover, if $c(\underline{\gamma}, S) \neq 0$ implies $S > 0$, then f is called a cusp form. The $\mathbb{C}$-vector space of modular forms of weight k with respect to $\rho_D^{(n)}$ and $\mathrm{Sp}_{2n}(\mathbb{Z})$ is denoted by $\mathrm{M}_k^{(n)}(D)$, the subspace of cusp forms by $\mathrm{S}_k^{(n)}(D)$.

The definition of the Petersson inner product naturally generalizes to

$$(f, g)^{(n)} = \int_{\Gamma^{(n)} \backslash \mathbb{H}_n} \langle f(Z), g(Z) \rangle \det(Y)^k \frac{\mathrm{d}X \mathrm{d}Y}{\det(Y)^{n+1}},$$

where $Z = X + iY$.

We can also define *Siegel Eisenstein series.* In fact, for any $v \in \mathbb{C}[D^n]$ that is invariant under the group $\Gamma_\infty^{(n)}$ and $k \in \mathbb{Z}$ with $k > n+1$ the series

$$E_{k,D,v}^{(n)} := \sum_{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_\infty^{(n)} \backslash \Gamma^{(n)}} \det(CZ + D)^{-k} \rho_D^{(n)}(M)^{-1} v$$

defines a modular form of weight k for $\rho_D^{(n)}$ (cf. [70, Theorem 1]). In particular let us denote

$$E_k^{(n)}(Z) := E_{k,D}^{(n)}(Z) := E_{k,D,\underline{0}}^{(n)}.$$

1.6 Theta series

We now want to define theta series weighted with harmonic polynomials. Again we generalize the definition in [30] to the vector-valued case in the same way as was done for $n = 1$ for example in [7].

Definition 1.6.1. A *harmonic form* of degree h in the matrix variable $X = (x_{ij})_{i=1,\dots,m;j=1,\dots,n}$ is a complex polynomial $P(X)$ with the properties

(i) $P(XA) = (\det A)^h P(X)$ for $A \in \mathbb{C}^{n\times n}$,

(ii) $\Delta P = \sum_{i,j} \frac{\partial^2}{(\partial x_{i,j})^2} P = 0$.

Let L be a positive-definite even lattice of even rank m with dual lattice L' and bilinear form $(\cdot,\cdot)$. We can choose an embedding $L \subset \mathbb{R}^m$ such that $(\cdot,\cdot)$ extends to the standard scalar product on $\mathbb{R}^m$. Let P be a harmonic form of degree $h \geq 0$. We define the *theta series* $\theta^{(n)}_{L,P}$ by

$$\theta^{(n)}_{L,P}(Z) := \sum_{\underline{\lambda}\in(L')^n} P(\underline{\lambda}) e^{\pi i \operatorname{tr}((\underline{\lambda},\underline{\lambda})Z)} \cdot e^{\underline{\lambda}+L},$$

where $(\underline{\lambda},\underline{\lambda}) = ((\lambda_i,\lambda_j))_{i,j=1}^n \in \operatorname{Mat}_n(\mathbb{R})$ for $\underline{\lambda} = (\lambda_1,\dots,\lambda_n)$. When P is identically 1 it is usually dropped from the notation. The Poisson summation formula implies

Theorem 1.6.2. *The theta series $\theta^{(n)}_{L,P}$ is a modular form of weight $m/2 + h$ with respect to $\rho^{(n)}_{L'/L}$ and $\operatorname{Sp}_{2n}(\mathbb{Z})$. If $h > 0$, then $\theta^{(n)}_{L,P}$ is a cusp form.*

Special cases of this theorem are very well-known. We will prove it using adelic theta series in the next section.

The *genus* G of an even lattice L is the set of isometry classes of lattices equivalent to L over $\mathbb{Z}_p$ for all primes p and over $\mathbb{R}$. It is uniquely determined by the signature (t_+, t_-) of L and its discriminant form L'/L (cf. [55]) and we denote this genus by $II_{t_+,t_-}(L'/L)$. Therefore, the positive-definite even lattices L of rank m with $L'/L \cong D$ form a genus denoted by $II_{m,0}(D)$.

We define the *genus theta series* of the genus $G = II_{m,0}(D)$ as

$$\theta^{(n)}_G := \mu(G)^{-1} |\operatorname{O}(D)|^{-1} \sum_{L\in G} \frac{1}{\#\operatorname{Aut}(L)} \sum_{\sigma\in\operatorname{Iso}(D,L'/L)} \sigma^{*(n)} \theta^{(n)}_L,$$

where

$$\mu(G) = \sum_{L\in G} \frac{1}{\#\operatorname{Aut}(L)}$$

is the mass of the genus G and the first sum ranges over the positive-definite even lattices in G.

We describe the theta series for $n = 1$ in more detail. Let H_m^h denote the space of harmonic polynomials in m variables homogeneous of degree h, i.e. harmonic forms of degree h in $(x_1, \dots, x_m)^T$. For a discriminant form D we define

$$\begin{aligned}\Theta_{m,k}(D) &\coloneqq \mathrm{span}\{\sigma^{*(1)}\theta_{L,P}^{(1)} \mid L \in \mathit{II}_{m,0}(D),\ P \in \mathrm{H}_m^{k-m/2},\ \sigma \in \mathrm{Iso}(D, L'/L)\} \\ &= \mathrm{span}\{\sigma_*^{(1)}\theta_{L,P}^{(1)} \mid L \in \mathit{II}_{m,0}(D),\ P \in \mathrm{H}_m^{k-m/2},\ \sigma \in \mathrm{Iso}(L'/L, D)\}\end{aligned}$$

and

$$\Theta_{m,k}(D)_0 \coloneqq \Theta_{m,k}(D) \cap \mathrm{S}_k^{(1)}(D).$$

Note that for $P \in \mathrm{H}_m^h$

$$\theta_{L,P} \coloneqq \theta_{L,P}^{(1)} = \sum_{\gamma \in L'/L} \theta_{\gamma,P} e^\gamma$$

and the component functions are given by $\theta_{\gamma,P}(\tau) = \sum_{\lambda \in \gamma + L} P(\lambda) e^{\pi i (\lambda,\lambda)\tau}$.

We can generate the space of harmonic polynomials using the so called *Gegenbauer polynomials* $G_m^h(s, n)$, which are defined by

$$\frac{1}{(1 - 2sX + nX^2)^{m/2-1}} = \sum_{h=0}^{\infty} G_m^h(s, n) \cdot X^h.$$

(cf. [28] and [38]). Then we obtain

Proposition 1.6.3. *The polynomial*

$$P_m^h(x, y) \coloneqq G_m^h(x^T y, \|x\|^2 \|y\|^2)$$

on $\mathbb{R}^m \times \mathbb{R}^m$ *is harmonic of degree* h *in both* x *and* y *when the other variable is fixed and* $P_m^h(Sx, Sy) = P_m^h(x, y)$ *for all* $S \in \mathrm{O}(m)$.

Proof. We denote

$$f(x) = \frac{1}{(1 - 2x^T yX + \|x\|^2 \|y\|^2 X^2)^{m/2-1}}$$

and compute

$$\begin{aligned}\frac{\partial^2}{(\partial x_i)^2} f(x) &= \frac{\partial}{\partial x_i} \frac{(-m/2 + 1)(-2y_i X + 2x_i \|y\|^2 X^2)}{(1 - 2x^T yX + \|x\|^2 \|y\|^2 X^2)^{m/2}} \\ &= \frac{(m/2 - 1)m/2(-2y_i X + 2x_i \|y\|^2 X^2)^2}{(1 - 2x^T yX + \|x\|^2 \|y\|^2 X^2)^{m/2+1}} \\ &\qquad - \frac{(m/2 - 1)2\|y\|^2 X^2}{(1 - 2x^T yX + \|x\|^2 \|y\|^2 X^2)^{m/2}}.\end{aligned}$$

Now

$$\sum_{i=1}^{m}(-2y_iX+2x_i\|y\|^2X^2)^2 = \sum_{i=1}^{m}(4y_i^2X^2-8x_iy_i\|y\|^2X^3+4x_i^2\|y\|^4X^4)$$
$$= 4\|y\|^2X^2(1-2x^TyX+\|x\|^2\|y\|^2X^2).$$

Hence

$$\Delta f(x) = \frac{(m-2)m\|y\|^2X^2}{(1-2x^TyX+\|x\|^2\|y\|^2X^2)^{m/2}} - \frac{(m-2)m\|y\|^2X^2)}{(1-2x^TyX+\|x\|^2\|y\|^2X^2)^{m/2}} = 0.$$

Furthermore, for a $\lambda \in \mathbb{R}$ we have

$$P_m^h(\lambda x, y) = G_m^h((\lambda x)^T y, \|\lambda x\|^2\|y\|) = G_m^h(\lambda x^T y, \lambda^2\|x\|^2\|y\|)$$

and

$$\sum_{h=0}^{\infty} G_m^h(\lambda s, \lambda^2 n)\cdot X^h = \frac{1}{(1-2\lambda sX+\lambda^2 nX^2)^{m/2-1}}$$
$$= \frac{1}{(1-2s(\lambda X)+n(\lambda X)^2)^{m/2-1}}$$
$$= \sum_{h=0}^{\infty} G_m^h(s,n)\cdot(\lambda X)^h$$
$$= \sum_{h=0}^{\infty} \lambda^h G_m^h(s,n)\cdot X^h$$

so that $P_m^h(\lambda x, y) = \lambda^h P_m^h(x,y)$. The fact that $P_m^h(Sx, Sy) = P_m^h(x,y)$ for all $S \in \mathrm{O}(m)$ follows from definition. □

Now let $Z = \left(\begin{smallmatrix} z_1 & z_2 \\ z_2 & z_4 \end{smallmatrix}\right) \in \mathbb{H}_2$ and $\underline{\lambda} = (\lambda, \mu) \in (L')^2$. We compute

$$e^{\pi i\,\mathrm{tr}((\underline{\lambda},\underline{\lambda})Z)} = e^{\pi i((\lambda,\lambda)z_1+2(\lambda,\mu)z_2+(\mu,\mu)z_4)}$$

and therefore

$$\frac{1}{2}\frac{\partial}{\partial z_2}e^{\pi i\,\mathrm{tr}((\underline{\lambda},\underline{\lambda})Z)}\Big|_{z_2=0} = \pi i(\lambda,\mu)e^{\pi i((\lambda,\lambda)z_1+(\mu,\mu)z_4)}$$
$$\frac{\partial^2}{\partial z_1\partial z_4}e^{\pi i\,\mathrm{tr}((\underline{\lambda},\underline{\lambda})Z)}\Big|_{z_2=0} = (\pi i)^2(\lambda,\lambda)(\mu,\mu)e^{\pi i((\lambda,\lambda)z_1+(\mu,\mu)z_4)}.$$

It follows that

$$G_m^h\left(\frac{1}{2}\frac{\partial}{\partial z_2}, \frac{\partial^2}{\partial z_1\partial z_4}\right)\theta_L^{(2)}(Z)\Big|_{z_2=0}$$
$$= (\pi i)^h \sum_{\gamma,\beta\in L'/L}\left[\sum_{\substack{\lambda\in\gamma+L\\ \mu\in\beta+L}} P_m^h(\lambda,\mu)e^{\pi i(\lambda,\lambda)z_1}e^{\pi i(\mu,\mu)z_2}\right]e^\gamma\otimes e^\beta$$

is in $\Theta_{m,m/2+h}(D) \otimes \Theta_{m,m/2+h}(D)$. In light of this we define on the space of C^∞ functions on $\mathbb{H}_2$ the operator

$$\partial_h : C^\infty(\mathbb{H}_2) \to C^\infty(\mathbb{H} \times \mathbb{H})$$
$$\partial_h f = G_m^h \left(\frac{1}{2} \frac{\partial}{\partial z_2}, \frac{\partial^2}{\partial z_1 \partial z_4} \right) f(Z)|_{z_2=0}.$$

Also let

$$\vartheta_{G,m/2+h} \coloneqq \partial_h \theta_G^{(2)} \in \Theta_{m,m/2+h}(D) \otimes \Theta_{m,m/2+h}(D).$$

Note that ∂_h is essentially the operator $\mathcal{D}_h$ defined by Eichler and Zagier in [28]. It is a special case of the operators studied in [38].

We define a scalar product $\mathrm{h}(\cdot,\cdot)$ on H_m^h by

$$\mathrm{h}(p,q) = \int_{B_1} \nabla p(x)^T \, \overline{\nabla q(x)} \mathrm{d}x.$$

An element $S \in \mathrm{SO}(m)$ naturally acts on H_m^h by $S.p = p(S^T \cdot)$ and it is well-known that this representation is irreducible (see for example [39, (0.9) and (5.7)]). Note that we have $\nabla S.p = S \cdot (\nabla p)(S^T \cdot)$ so that for any $p, q \in \mathrm{H}_m^h$ we obtain

$$\begin{aligned}\mathrm{h}(S.p, S.q) &= \int_{B_1} \nabla p(S^T x)^T \, \overline{\nabla q(S^T x)} \mathrm{d}x = \int_{B_1} [(\nabla p)(S^T x)]^T S^T S \, \overline{(\nabla q)(S^T x)} \mathrm{d}x \\ &= \int_{S^T B_1} \nabla p(x)^T \, \overline{\nabla q(x)} \mathrm{d}x = \mathrm{h}(p,q),\end{aligned}$$

where we substituted Sx for x in the last step and used the fact that $S^T S = I$. This implies

Proposition 1.6.4. *Let m be even and $h > 0$ and let $(P_1, \ldots, P_r)$ be any orthonormal basis of H_m^h with respect to $\mathrm{h}(\cdot,\cdot)$. Then $P_m^h(x,y)$ is up to a non-zero constant equal to*

$$\sum_{i=1}^r P_i(x) \overline{P_i}(y).$$

Proof. We define a map $\phi : \mathrm{H}_m^h \to \mathrm{H}_m^h$ by

$$\phi(p)(x) = \mathrm{h}(p, P_m^h(\cdot, \overline{x})).$$

We find that

$$(S.\phi(p))(x) = \mathrm{h}(p, P_m^h(\cdot, \overline{S^T x})) = \mathrm{h}(p, P_m^h(S\cdot, \overline{x})) = \mathrm{h}(S.p, P_m^h(\cdot, \overline{x})) = \phi(S.p)(x)$$

and so by Schur's lemma, ϕ must be a multiple of the identity. Since P_m^h is non-zero, this scalar is non-zero. Clearly the identity map is given by

$$p \mapsto \sum_{i=1}^r \mathrm{h}(p, P_i) P_i.$$

□

1.7 The adelic Weil representation and the Siegel–Weil formula

We have now seen the Weil representation of $\mathrm{Sp}_{2n}(\mathbb{Z})$. It is in fact a subrepresentation of a representation of $\mathrm{Sp}_{2n}(\mathbb{A})$ introduced by Weil in [69]. Even though the results in this work are concerned with the Weil representation of $\mathrm{Mp}_2(\mathbb{Z})$, we want to describe the adelic setting as well, in order to use its rich toolkit. In [70] the famous Siegel–Weil formula was proven, which essentially states that taking a certain average of a theta series is equal to a value of the Eisenstein series. Translating this back to the classical setting we can show that the genus theta series $\theta_G^{(n)}$ defined in the previous section is equal to the Eisenstein series $E_{k,D}^{(n)}$.

Consider $\mathbb{Z}^{2n}$ together with the alternating bilinear form given by $(x, y) \mapsto x^T J_n y$ and let Sp_{2n} be the corresponding symplectic $\mathbb{Z}$-group scheme. We denote by $P = AN \subset \mathcal{G} = \mathrm{Sp}_{2n}$ the standard Siegel parabolic subgroup, so that we have

$$A(R) = \left\{ a(u) = \begin{pmatrix} u & 0 \\ 0 & (u^T)^{-1} \end{pmatrix} \mid u \in \mathrm{GL}_n(R) \right\},$$
$$N(R) = \left\{ n(s) = \begin{pmatrix} I_n & s \\ 0 & I_n \end{pmatrix} \mid s \in \mathrm{Sym}_n(R) \right\}$$

for any ring $R \supset \mathbb{Z}$. Let V be a quadratic space over $\mathbb{Q}$ of even dimension m with non-degenerate symmetric bilinear form $(\cdot, \cdot)$ and $V(R) = V \otimes_{\mathbb{Q}} R$ for any ring $R \supset \mathbb{Q}$. Let $\mathrm{O}(V)$ be the corresponding orthogonal group scheme. Then $(\mathcal{G}, \mathcal{H}) = (\mathrm{Sp}_{2n}, \mathrm{O}(V))$ is a reductive dual pair.

Let $Z = X + iY \in \mathbb{H}_n$ and $U \in \mathrm{Pos}_n(\mathbb{R})$ with $UU^T = Y$. Because of $n(X)a(U) \cdot iI_n = X + iY$, the subgroup $P(\mathbb{R}) \subset \mathcal{G}(\mathbb{R})$ already acts transitively on $\mathbb{H}_n$ and we denote $g_Z := n(X)a(U)$. Thus, we can identify the upper half-space with $\mathcal{G}(\mathbb{R})/K_\infty$, where $K_\infty \subset \mathcal{G}(\mathbb{R})$ is the stabilizer of iI_n and given by

$$K_\infty = \left\{ \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \mid u + iv \in U(n) \right\}.$$

Note that $u + iv \in U(n)$ if and only if $u^T u + v^T v = I_n$ and $u^T v = v^T u$. Now let $f \in \mathrm{M}_k^{(n)}(D)$ for some discriminant form D of even signature. If we set

$$\phi_f(g) = f|_k[g](iI_n), \qquad g \in \mathcal{G}(\mathbb{R}),$$

then for $M \in \mathrm{Sp}_{2n}(\mathbb{Z})$ we have

$$\phi_f(Mg) = \rho_D^{(n)}(M)\phi_f(g).$$

For a prime p let $\mathbb{Q}_p$ denote the completion of $\mathbb{Q}$ at p and $\mathbb{Z}_p$ the ring of integers of $\mathbb{Q}_p$. Let $\hat{\mathbb{Z}} = \prod_{p<\infty} \mathbb{Z}_p$ and let $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$ be the ring of adeles of $\mathbb{Q}$. The idea is now to identify $\rho_D^{(n)}$ with a representation of $\mathcal{G}(\mathbb{A}_f)$, so that f corresponds to a function on $\mathcal{G}(\mathbb{A})$ invariant under $\mathcal{G}(\mathbb{Q})$.

Let $\mathscr{S}(V(\mathbb{A})^n)$ be the space of Schwartz functions on $V(\mathbb{A})^n$. For an additive character $\psi : \mathbb{A} \to \mathbb{C}^\times$ we will define a representation $\omega = \omega_{V,\psi}$ of $\mathcal{G}(\mathbb{A}) \times \mathcal{H}(\mathbb{A})$ on this space that we will call the *adelic Weil representation*. It generalizes the representations described in Section 1.4. Let $\operatorname{disc}(V) \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ be the discriminant of V defined to be

$$\operatorname{disc}(V) \coloneqq (-1)^{m/2} \det((x_i, x_j))_{i,j=1}^m$$

for any $\mathbb{Q}$-basis $\{x_1, \ldots, x_m\}$ of V and let $\chi_V : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ be the quadratic character that corresponds to the quadratic extension $\mathbb{Q}(\sqrt{\operatorname{disc}(V)})/\mathbb{Q}$, i.e. $\chi_V(x) = (x, \operatorname{disc}(V))$, where $(\cdot,\cdot)$ is the Hilbert symbol. We denote by $|\cdot| : \mathbb{A}^\times \to \mathbb{R}_{>0}$ the normalized absolute value and define $(\underline{x}, \underline{y}) \coloneqq ((x_i, y_j))_{i,j=1}^n \in \operatorname{Mat}_n(\mathbb{A})$ for $\underline{x} = (x_1, \ldots, x_n) \in V(\mathbb{A})^n$ and $\underline{y} = (y_1, \ldots, y_n) \in V(\mathbb{A})^n$. Finally, denote by $\hat{\varphi}$ the Fourier transform of φ using the self-dual Haar measure on $V(\mathbb{A})^n$ with respect to ψ, i.e.

$$\hat{\varphi}(\underline{x}) = \int_{V(\mathbb{A})^n} \varphi(\underline{y})\psi(\operatorname{tr}(\underline{x}, \underline{y}))d\underline{y}.$$

Let us now fix the standard additive character $\psi : \mathbb{A} \to \mathbb{C}^\times$ whose archimedean component is given by $\psi_\infty : \mathbb{R} \to \mathbb{C}^\times,\ x_\infty \mapsto e(x_\infty)$ and the finite components by $\psi_p : \mathbb{Q}_p \to \mathbb{C}^\times,\ x_p \mapsto e(-x_p')$, where $x_p' \in \mathbb{Q}/\mathbb{Z}$ is the principal part of x_p. The Weil representation $\omega = \omega_{V,\psi}$ is the representation of $\mathcal{G}(\mathbb{A}) \times \mathcal{H}(\mathbb{A})$ on the space of Schwartz functions $\mathscr{S}(V(\mathbb{A})^n)$ that is uniquely determined by the following equations: Let $\varphi \in \mathscr{S}(V(\mathbb{A})^n)$ and $\underline{x} \in V(\mathbb{A})^n$, then

$$\begin{aligned}
\omega(a(u))\varphi(\underline{x}) &= \chi_V(\det(u))|\det(u)|^{m/2}\varphi(\underline{x} \cdot u), && a(u) \in A(\mathbb{A}),\\
\omega(n(s))\varphi(\underline{x}) &= \psi(\tfrac{1}{2}\operatorname{tr}(s(\underline{x}, \underline{x})))\varphi(\underline{x}), && n(s) \in N(\mathbb{A}),\\
\omega(J_n)\varphi(\underline{x}) &= \hat{\varphi}(\underline{x}), &&\\
\omega(h)\varphi(\underline{x}) &= \varphi(h^{-1} \cdot \underline{x}), && h \in H(\mathbb{A}).
\end{aligned}$$

Note that for the local components ω_p we define $\omega_p(J_n)\varphi_p(\underline{x}) = \gamma_p^n \hat{\varphi}_p(\underline{x})$, where γ_p is a specific eighth root of unity and

$$\gamma_\infty \prod_{p<\infty} \gamma_p = 1.$$

If (t_+, t_-) is the signature of the quadratic space V, then $\gamma_\infty = e((t_+ - t_-)/8)$. If L is an even lattice in V, then $\rho_{L'/L}$ or rather its dual $\overline{\rho}_{L'/L}$ can be viewed as a subrepresentation of $\omega_{V,\psi}$. More precisely, let L be a positive-definite even lattice of even rank m and set $V := L \otimes \mathbb{Q}$. In this case $\gamma_p = \gamma_p(L'/L)^{-1}$ for all primes p. Consider the subspace $\mathscr{S}_L$ of Schwartz functions in $\mathscr{S}(V(\mathbb{A}_f)^n)$ which are supported on $(L')^n \otimes \hat{\mathbb{Z}}$ and which are constant on cosets of $L^n \otimes \hat{\mathbb{Z}}$. There is an isomorphism

$$\iota : \mathbb{C}[(L'/L)^n] \to \mathscr{S}_L$$

given by mapping a basis element $e^{\underline{\mu}}$ of $\mathbb{C}[(L'/L)^n]$ to $\varphi_{\underline{\mu}} = \bigotimes_{p<\infty} \varphi_p$, where $\varphi_p \in \mathscr{S}(V(\mathbb{Q}_p)^n)$ is the characteristic function of $\underline{\mu} + (L \otimes \mathbb{Z}_p)^n$. We obtain (cf. [74, section 2.1.3])

Proposition 1.7.1. *For any $M \in \Gamma^{(n)}$ we have*

$$\omega_f(M) \circ \iota = \iota \circ \overline{\rho}^{(n)}_{L'/L}(M).$$

Proof. We need to show that

$$\begin{aligned} \omega_f(n(S))\iota(e^{\underline{\mu}}) &= \iota(\overline{\rho}^{(n)}_{L'/L}(n(S))e^{\underline{\mu}}) \text{ and} \\ \omega_f(J_n)\iota(e^{\underline{\mu}}) &= \iota(\overline{\rho}^{(n)}_{L'/L}(J_n)e^{\underline{\mu}}) \end{aligned}$$

for all symmetric S. But the first identity is clear from the choice of ψ. For the second we compute

$$\begin{aligned} (\omega_f(J_n)\iota(e^{\underline{\mu}}))(\underline{x}_f) &= e(-n\,\mathrm{sign}(L'/L)/8) \int_{\underline{\mu}+(L\otimes\hat{\mathbb{Z}})^n} \psi_f(\mathrm{tr}(\underline{x}_f, \underline{y}_f))\mathrm{d}\underline{y}_f \\ &= e(-n\,\mathrm{sign}(L'/L)/8) \int_{(L\otimes\hat{\mathbb{Z}})^n} \psi_f(\mathrm{tr}(\underline{x}_f, \underline{\mu} + \underline{y}_f))\mathrm{d}\underline{y}_f \\ &= e(-n\,\mathrm{sign}(L'/L)/8)\psi_f(\mathrm{tr}(\underline{x}_f, \underline{\mu})) \int_{(L\otimes\hat{\mathbb{Z}})^n} \psi_f(\mathrm{tr}(\underline{x}_f, \underline{y}_f))\mathrm{d}\underline{y}_f. \end{aligned}$$

Clearly if $\underline{x}_f \in (L' \otimes \hat{\mathbb{Z}})^n$ then

$$\int_{(L\otimes\hat{\mathbb{Z}})^n} \psi_f(\mathrm{tr}(\underline{x}_f, \underline{y}_f))\mathrm{d}\underline{y}_f = \int_{(L\otimes\hat{\mathbb{Z}})^n} \mathrm{d}\underline{y}_f = \mathrm{vol}((L \otimes \hat{\mathbb{Z}})^n).$$

If $\underline{x}_f \notin (L' \otimes \hat{\mathbb{Z}})^n$, then there exists some $\underline{\lambda} \in (L \otimes \hat{\mathbb{Z}})^n$ such that $\psi_f(\mathrm{tr}(\underline{x}_f, \underline{\lambda})) \neq 1$. But then

$$\begin{aligned} \psi_f(\mathrm{tr}(\underline{x}_f, \underline{\lambda})) \int_{(L\otimes\hat{\mathbb{Z}})^n} \psi_f(\mathrm{tr}(\underline{x}_f, \underline{y}_f))\mathrm{d}\underline{y}_f &= \int_{(L\otimes\hat{\mathbb{Z}})^n} \psi_f(\mathrm{tr}(\underline{x}_f, \underline{\lambda} + \underline{y}_f))\mathrm{d}\underline{y}_f \\ &= \int_{(L\otimes\hat{\mathbb{Z}})^n} \psi_f(\mathrm{tr}(\underline{x}_f, \underline{y}_f))\mathrm{d}\underline{y}_f. \end{aligned}$$

And so

$$(1 - \psi_f(\mathrm{tr}(\underline{x}_f, \underline{\lambda}))) \int_{(L\otimes\hat{\mathbb{Z}})^n} \psi_f(\mathrm{tr}(\underline{x}_f, \underline{y}_f))\mathrm{d}\underline{y}_f = 0,$$

which implies

$$\int_{(L\otimes\hat{\mathbb{Z}})^n} \psi_f(\mathrm{tr}(\underline{x}_f, \underline{y}_f))\mathrm{d}\underline{y}_f = 0.$$

Using the fact that $\psi_f(\mathrm{tr}(\underline{x}_f, \underline{\mu}))$ is constant on cosets of $(L \otimes \hat{\mathbb{Z}})^n$ we obtain

$$(\omega_f(J_n)\iota(e^{\underline{\mu}}))(\underline{x}_f) = e(-n\,\mathrm{sign}(L'/L)/8)\mathrm{vol}((L\otimes\hat{\mathbb{Z}})^n) \sum_{\underline{\beta}\in(L'/L)^n} \psi_f(\mathrm{tr}(\underline{\beta}, \underline{\mu}))\iota(e^{\underline{\beta}})(\underline{x}_f)$$

with $\psi_f(\mathrm{tr}(\underline{\beta}, \underline{\mu})) = e(-(\underline{\beta}, \underline{\mu}))$. So it remains to compute $\mathrm{vol}((L \otimes \hat{\mathbb{Z}})^n)$. Let $\varphi = \iota(e^{\underline{0}})$. Then we have

$$\hat{\varphi} = \mathrm{vol}((L\otimes\hat{\mathbb{Z}})^n) \sum_{\underline{\beta}\in(L'/L)^n} \iota(e^{\underline{\beta}})$$

and

$$\begin{aligned}\hat{\hat{\varphi}} &= \mathrm{vol}((L\otimes\hat{\mathbb{Z}})^n)^2 \sum_{\underline{\beta},\underline{\mu}\in(L'/L)^n} e(-(\underline{\mu}, \underline{\beta}))\iota(e^{\underline{\mu}}) \\ &= \mathrm{vol}((L\otimes\hat{\mathbb{Z}})^n)^2 |L'/L|^n \iota(e^{\underline{0}}) \\ &= \mathrm{vol}((L\otimes\hat{\mathbb{Z}})^n)^2 |L'/L|^n \varphi.\end{aligned}$$

But by our choice of normalization $\hat{\hat{\varphi}}(\underline{x}_f) = \varphi(-\underline{x}_f) = \varphi(\underline{x}_f)$ and so

$$\mathrm{vol}((L\otimes\hat{\mathbb{Z}})^n) = \frac{1}{\sqrt{|L'/L|^n}},$$

which proves the proposition. □

We want to define theta series and Eisenstein series in the adelic setting. These will be generalizations of the corresponding classical objects. We assume from now on that V is positive definite and so $(t_+, t_-) = (m, 0)$.

Definition 1.7.2. For $\varphi \in \mathscr{S}(V(\mathbb{A})^n)$ define the *theta series*

$$\theta(g, h; \varphi) := \sum_{\underline{x}\in V^n} \omega(g)\varphi(h^{-1}\underline{x}), \qquad g \in \mathcal{G}(\mathbb{A}),\ h \in \mathcal{H}(\mathbb{A}).$$

Then $\theta(g, h; \varphi)$ is automorphic on both $\mathcal{G}$ and $\mathcal{H}$ (i.e. invariant under $\mathcal{G}(\mathbb{Q}) \times \mathcal{H}(\mathbb{Q})$) by Poisson summation.

Definition 1.7.3. For $\varphi \in \mathscr{S}(V(\mathbb{A})^n)$ define the *Siegel Eisenstein series*

$$E(g,s;\varphi) := \sum_{\gamma \in P(\mathbb{Q})\backslash \mathcal{G}(\mathbb{Q})} \Phi_\varphi(\gamma g, s), \qquad g \in G(\mathbb{A}),\ s \in \mathbb{C},$$

where

$$\mathscr{S}(V(\mathbb{A})^n) \to \mathrm{Ind}_{P(\mathbb{A})}^{\mathcal{G}(\mathbb{A})}(\chi_V |\cdot|^s), \qquad \varphi \mapsto \Phi_\varphi(g,s) := (\omega(g)\varphi)(0)\cdot |\det u(g)|^{s-s_0}$$

is the *standard Siegel–Weil section* and

$$s_0 := \frac{m-(n+1)}{2}.$$

Here we write $g = na(u)k$ under the Iwasawa decomposition $\mathcal{G}(\mathbb{A}) = N(\mathbb{A})A(\mathbb{A})K$ for K the standard maximal open compact subgroup of $\mathcal{G}(\mathbb{A})$, and the quantity $|\det u(g)| := |\det u|$ is well-defined.

The Eisenstein series converges absolutely for $\mathrm{Re}(s) > \frac{n+1}{2}$ and thus defines an automorphic form for $\mathcal{G}(\mathbb{Q})$.

In order to retrieve the classical theta series and Eisenstein series of Sections 1.5 and 1.6 from the adelic versions, we will need to choose a specific φ. Of particular importance will be the standard Gaussian $e^{-\pi\,\mathrm{tr}(\underline{x},\underline{x})}$. We have

Lemma 1.7.4. *Let P be a harmonic form of degree h and let $\varphi(\underline{x}) = P(\underline{x})e^{-\pi\,\mathrm{tr}(\underline{x},\underline{x})}$. Furthermore, let $g \in \mathcal{G}(\mathbb{R})$ with $g \cdot iI_n = Z$. Then*

$$(\omega_\infty(g)\varphi)(\underline{x}) = j(g, iI_n)^{-m/2-h} P(\underline{x}) e^{\pi i\,\mathrm{tr}((\underline{x},\underline{x})Z)}.$$

Proof. First note that the claim is true for $g = I$. We show that for $g \in \mathcal{G}(\mathbb{R})$ with $g \cdot iI_n = Z$ and g' equal to $a(U)$, $n(S)$ or J we have

$$\omega_\infty(g') j(g, iI_n)^{-m/2-h} P(\underline{x}) e^{\pi i\,\mathrm{tr}((\underline{x},\underline{x})Z)} = j(g', Z)^{-m/2-h} j(g, iI_n)^{-m/2-h} P(\underline{x}) e^{\pi i\,\mathrm{tr}((\underline{x},\underline{x})g'\cdot Z)}. \quad (1.7.1)$$

Then the claim follows from the cocylce relation $j(g', Z)j(g, iI_n) = j(g'g, iI_n)$ and the fact that $\mathcal{G}(\mathbb{R})$ is generated by elements of the form $a(U)$, $n(S)$ and J.
We have

$$\begin{aligned}\omega_\infty(a(U))P(\underline{x})e^{\pi i\,\mathrm{tr}((\underline{x},\underline{x})Z)} &= \chi_V(\det(U))|\det(U)|_\infty^{m/2} P(\underline{x}U) e^{\pi i\,\mathrm{tr}((\underline{x}U,\underline{x}U)Z)} \\ &= \det(U)^{m/2+h} P(\underline{x}) e^{\pi i\,\mathrm{tr}((\underline{x},\underline{x})UZU^T)},\end{aligned}$$

where we have used that $\chi_V(x) = \operatorname{sgn}(x)^{m/2}$ for $x \in \mathbb{R}$ and that tr is invariant under cyclic permutations. Furthermore,

$$\begin{aligned}\omega_\infty(n(S))P(\underline{x})e^{\pi i \operatorname{tr}((\underline{x},\underline{x})Z)} &= \psi_\infty(\tfrac{1}{2}\operatorname{tr}(S(\underline{x},\underline{x})))P(\underline{x})e^{\pi i \operatorname{tr}((\underline{x},\underline{x})Z)} \\ &= P(\underline{x})e^{\pi i \operatorname{tr}(S(\underline{x},\underline{x})+(\underline{x},\underline{x})Z)} \\ &= P(\underline{x})e^{\pi i \operatorname{tr}((\underline{x},\underline{x})(Z+S))}.\end{aligned}$$

Finally, for $g' = J$ we show (1.7.1) for $Z = iY$. Since both sides of the equation are analytic in Z, by the identity theorem the general case follows. We have

$$\begin{aligned}&\omega_\infty(J)P(\underline{x})e^{\pi i \operatorname{tr}((\underline{x},\underline{x})Z)} \\ &\quad= e(nm/8)\int_{V(\mathbb{R})^n} P(\underline{y})e^{\pi i \operatorname{tr}((\underline{y},\underline{y})Z)}e^{2\pi i \operatorname{tr}((\underline{x},\underline{y}))}\mathrm{d}\underline{y} \\ &\quad= e(nm/8)e^{\pi i \operatorname{tr}(-(\underline{x},\underline{x})Z^{-1})}\int_{V(\mathbb{R})^n} P(\underline{y})e^{\pi i \operatorname{tr}((\underline{y}+\underline{x}Z^{-1},\underline{y}+\underline{x}Z^{-1})Z)}\mathrm{d}\underline{y} \\ &\quad= e(nm/8)e^{\pi i \operatorname{tr}(-(\underline{x},\underline{x})Z^{-1})}\int_{V(\mathbb{R})^n} P(\underline{y})e^{-\pi \operatorname{tr}((\underline{y}-i\underline{x}Y^{-1},\underline{y}-i\underline{x}Y^{-1})Y)}\mathrm{d}\underline{y}\end{aligned}$$

Let us for now assume that $\underline{x}$ is purely imaginary. Then the second integral equals

$$\int_{V(\mathbb{R})^n} P(\underline{y}+i\underline{x}Y^{-1})e^{-\pi \operatorname{tr}((\underline{y},\underline{y})Y)}\mathrm{d}\underline{y}$$

by a change of variables. Since $Y \in \operatorname{Pos}_n(\mathbb{R})$, there exists a $U \in \operatorname{Pos}_n(\mathbb{R})$ with $U^2 = Y$. Substituting $\underline{y}$ by $\underline{y}U^{-1}$ we get

$$\begin{aligned}&\int_{V(\mathbb{R})^n} P(\underline{y}U^{-1}+i\underline{x}Y^{-1})e^{-\pi \operatorname{tr}((\underline{y},\underline{y}))}\det(U)^{-m}\mathrm{d}\underline{y} \\ &\qquad= \det(U)^{-m-h}\int_{V(\mathbb{R})^n} P(\underline{y}+i\underline{x}U^{-1})e^{-\pi \operatorname{tr}((\underline{y},\underline{y}))}\mathrm{d}\underline{y} \\ &\qquad= \det(U)^{-m-h}P(i\underline{x}U^{-1}) \\ &\qquad= \det(Y)^{-m/2-h}i^h P(\underline{x}),\end{aligned}$$

where we have used that harmonic forms are invariant under Gaussian transformation (see [47]). Again, by the identity theorem the equality follows for all $\underline{x} \in V(\mathbb{C})^n$, in particular, for $\underline{x} \in V(\mathbb{R})^n$. We obtain

$$\begin{aligned}\omega_\infty(J)P(\underline{x})e^{\pi i \operatorname{tr}((\underline{x},\underline{x})iY)} &= e(nm/8)e^{\pi i \operatorname{tr}(-(\underline{x},\underline{x})(iY)^{-1})}\det(Y)^{-m/2-h}i^h P(\underline{x}) \\ &= \det(-iY)^{-m/2-h}P(\underline{x})e^{\pi i \operatorname{tr}(-(\underline{x},\underline{x})(iY)^{-1})}.\end{aligned}$$

□

With the previous lemma we can see the relation between the theta series defined in this section and the classical theta series defined earlier.

Proposition 1.7.5. *Let L be a positive-definite even lattice of even rank m, let $\iota : \mathbb{C}[(L'/L)^n] \to \mathscr{S}_L$ be as above and let P be a harmonic form of degree h and $\varphi_\infty(\underline{x}) = P(\underline{x})e^{-\pi \operatorname{tr}(\underline{x},\underline{x})}$. We set $\varphi = \iota(v) \otimes \varphi_\infty$ for some $v \in \mathbb{C}[(L'/L)^n]$. For any $g \in \mathcal{G}(\mathbb{R})$ with $g \cdot iI_n = Z$ we have*

$$j(g, iI_n)^{m/2+h}\theta(g, 1; \varphi) = \langle \theta^{(n)}_{L,P}(Z), \overline{v}\rangle.$$

Proof. Since both the left and right hand side are linear in v, it suffices to prove the identity for $v = e^{\underline{\mu}}$ for some $\underline{\mu} \in (L'/L)^n$. So let φ_p be the characteristic function of $\underline{\mu} + (L \otimes \mathbb{Z}_p)^n$. Since $g \in \mathcal{G}(\mathbb{R})$, it acts trivially in ω_f and so by the previous lemma and the fact that $\bigcap_{p<\infty}(L \otimes \mathbb{Z}_p \cap V) = L$ we have

$$\begin{aligned} j(g, iI_n)^{m/2+h}\theta(g, 1; \varphi) &= j(g, iI_n)^{m/2+h} \sum_{\underline{x}\in V^n} \omega(g)\varphi(\underline{x}) \\ &= \sum_{\underline{x}\in\underline{\mu}+L^n} P(\underline{x})e^{\pi i \operatorname{tr}((\underline{x},\underline{x})Z)} \\ &= \langle \theta^{(n)}_{L,P}(Z), e^{\underline{\mu}}\rangle. \end{aligned}$$

□

Now we can show that the theta series defined in the previous section are modular forms for the Weil representation.

Proof of Theorem 1.6.2. Let $\varphi_\infty(\underline{x}) = P(\underline{x})e^{-\pi \operatorname{tr}(\underline{x},\underline{x})}$ for some harmonic form P of degree h and let $M \in \Gamma^{(n)}$ and $g \in \mathcal{G}(\mathbb{R})$ such that $g \cdot iI_n = Z$. Let us denote by M_∞ the element in $\mathcal{G}(\mathbb{A})$ that is equal to the identity in all finite places and equal to M in $\mathcal{G}(\mathbb{R})$ and $M_f = M_\infty^{-1}M$. We set $k = m/2 + h$. Then

$$\begin{aligned} \theta^{(n)}_{L,P}|_k[M](Z) &= j(M, Z)^{-k} j(Mg, iI_n)^k \sum_{\underline{\mu}\in(L'/L)^n} \theta(M_\infty g, 1; \varphi_\infty \otimes \iota(e^{\underline{\mu}}))e^{\underline{\mu}} \\ &= j(g, iI_n)^k \sum_{\underline{\mu}\in(L'/L)^n} \theta(MM_f^{-1}g, 1; \varphi_\infty \otimes \iota(e^{\underline{\mu}}))e^{\underline{\mu}} \\ &= j(g, iI_n)^k \sum_{\underline{\mu}\in(L'/L)^n} \theta(M_f^{-1}g, 1; \varphi_\infty \otimes \iota(e^{\underline{\mu}}))e^{\underline{\mu}} \end{aligned}$$

since $\vartheta(\cdot, 1; \varphi)$ is invariant under $\mathcal{G}(\mathbb{Q})$. We get

$$\begin{aligned}
j(g, iI_n)^k & \sum_{\underline{\mu}\in(L'/L)^n} \sum_{\underline{x}\in V^n} \omega(M_f^{-1}g)(\varphi_\infty \otimes \iota(e^{\underline{\mu}}))(\underline{x})e^{\underline{\mu}} \\
&= j(g, iI_n)^k \sum_{\underline{\mu}\in(L'/L)^n} \sum_{\underline{x}\in V^n} (\omega_\infty(g)\varphi_\infty \otimes \omega_f(M^{-1})\iota(e^{\underline{\mu}}))(\underline{x})e^{\underline{\mu}} \\
&= j(g, iI_n)^k \sum_{\underline{\mu}\in(L'/L)^n} \sum_{\underline{x}\in V^n} (\omega_\infty(g)\varphi_\infty \otimes \iota(\overline{\rho}^{(n)}_{L'/L}(M^{-1})e^{\underline{\mu}}))(\underline{x})e^{\underline{\mu}} \\
&= \sum_{\underline{\mu}\in(L'/L)^n} \langle \theta^{(n)}_{L,P}(Z), \rho^{(n)}_{L'/L}(M^{-1})e^{\underline{\mu}}\rangle e^{\underline{\mu}},
\end{aligned}$$

where we used Propositions 1.7.1 and 1.7.5. Now it follows

$$\begin{aligned}
\sum_{\underline{\mu}\in(L'/L)^n} \langle \theta^{(n)}_{L,P}(Z), \rho^{(n)}_{L'/L}(M^{-1})e^{\underline{\mu}}\rangle e^{\underline{\mu}} &= \sum_{\underline{\mu}\in(L'/L)^n} \langle \rho^{(n)}_{L'/L}(M)\theta^{(n)}_{L,P}(Z), e^{\underline{\mu}}\rangle e^{\underline{\mu}} \\
&= \rho^{(n)}_{L'/L}(M)\theta^{(n)}_{L,P}(Z).
\end{aligned}$$

□

Similar to the theta series we also see how the Eisenstein series in the adelic setting relate to the classical Eisenstein series.

Proposition 1.7.6. *Let L be a positive-definite even lattice of even rank m, let $\iota : \mathbb{C}[(L'/L)^n] \to \mathscr{S}_L$ be as above and let $\varphi_\infty(\underline{x}) = e^{-\pi \operatorname{tr}(\underline{x},\underline{x})}$ be the standard Gaussian. We set $\varphi = \iota(v) \otimes \varphi_\infty$ for some $v \in \mathbb{C}[(L'/L)^n]$. For any $g \in \mathcal{G}(\mathbb{R})$ with $g \cdot iI_n = Z$ we have*

$$j(g, iI_n)^{m/2} E(g, s_0; \varphi) = \langle E^{(n)}_{m/2,L'/L}(Z), \overline{v}\rangle.$$

Proof. Once again we need only prove the identity for $v = e^{\underline{\mu}}$ for some $\underline{\mu} \in D^n$. It is not difficult to prove that $P(\mathbb{Q})\backslash\mathcal{G}(\mathbb{Q}) \cong \Gamma^{(n)}_\infty\backslash\Gamma^{(n)}$. We write

$$\Phi_\varphi(g, s) = \Phi_f(g, s) \otimes \Phi_\infty(g, s)$$

with

$$\Phi_f(g, s) = \bigotimes_{p \text{ prime}} \Phi_p(g, s).$$

By Lemma 1.7.4 for $M \in \Gamma^{(n)}$ we have

$$\Phi_\infty(Mg, s_0) = j(Mg, iI_n)^{-m/2} = j(M, Z)^{-m/2} j(g, iI_n)^{-m/2}.$$

Note that for $v \in \mathbb{C}[D^n]$, $\underline{\beta} \in D^n$ and $\underline{x} \in \underline{\beta} + L^n$ we have $\iota(v)(\underline{x}) = \langle v, e^{\underline{\beta}} \rangle$. Furthermore, $\omega(g)$ acts trivially on $\iota(e^{\underline{\mu}})$ since $g \in \mathcal{G}(\mathbb{R})$ and therefore

$$\begin{aligned} j(g, iI_n)^{m/2} \cdot E(g, s_0; \varphi_\infty \otimes \iota(e^{\underline{\mu}})) &= \sum_{M \in \Gamma_\infty^{(n)} \backslash \Gamma^{(n)}} j(M, Z)^{-m/2} \omega_f(M) \iota(e^{\underline{\mu}})(0) \\ &= \sum_{M \in \Gamma_\infty^{(n)} \backslash \Gamma^{(n)}} j(M, Z)^{-m/2} \langle \overline{\rho}_{L'/L}^{(n)}(M) e^{\underline{\mu}}, e^{\underline{0}} \rangle \\ &= \sum_{M \in \Gamma_\infty^{(n)} \backslash \Gamma^{(n)}} j(M, Z)^{-m/2} \langle \rho_{L'/L}^{(n)}(M^{-1}) e^{\underline{0}}, e^{\underline{\mu}} \rangle \\ &= \langle E_{m/2, L'/L}^{(n)}, e^{\underline{\mu}} \rangle, \end{aligned}$$

where we used proposition 1.7.1. □

The average of the theta series over the orthogonal group is equal to the Eisenstein series at s_0.

Theorem 1.7.7 (Siegel–Weil formula (cf. [70] and [44]))**.** *Let V be a positive-definite quadratic space of even rank m over $\mathbb{Q}$ and let $\varphi \in \mathscr{S}(V(\mathbb{A})^n)$ with $m > 2n + 2$. Then $E(g, s; \varphi)$ is holomorphic at s_0 and*

$$\int_{\mathcal{H}(\mathbb{Q}) \backslash \mathcal{H}(\mathbb{A})} \theta(g, h; \varphi) \mathrm{d}h = E(g, s_0; \varphi).$$

Here the Haar measure $\mathrm{d}h$ is normalized so that $\mathrm{vol}(\mathcal{H}(\mathbb{Q}) \backslash \mathcal{H}(\mathbb{A})) = 1$.

Now we can use Theorem 1.7.7 to prove that the genus theta series defined in section 1.6 is equal to the Eisenstein series defined in section 1.5.

Theorem 1.7.8. *Let L be a positive-definite even lattice of even rank m. Let G be the genus of L. Then*

$$\theta_G^{(n)} = E_{m/2, L'/L}^{(n)}.$$

for $m > 2n + 2$.

Proof. We generalize the proof in [46]. Let $V := L \otimes_{\mathbb{Z}} \mathbb{Q}$, $D = L'/L$ and $\underline{\mu} = (\mu_1, \ldots, \mu_n) \in D^n$. Let $g \in \mathcal{G}(\mathbb{R})$ with $g \cdot iI_n = Z$ and $\varphi_\infty(\underline{x}) = e^{-\pi \operatorname{tr}(\underline{x}, \underline{x})}$ be the standard Gaussian function and set $\varphi = \varphi_\infty \otimes \iota(e^{\underline{\mu}})$. We show that for this choice of φ the left hand side of Theorem 1.7.7 is equal to $j(g, iI_n)^{-m/2} \langle \theta_G^{(n)}, e^{\underline{\mu}} \rangle$. By Proposition 1.7.6, the right hand side is equal to $j(g, iI_n)^{-m/2} \langle E_{m/2, L'/L}^{(n)}, e^{\underline{\mu}} \rangle$, which proves the theorem.

Let $S \subset \mathcal{H}(\mathbb{A})$ be the stabilizer of L. Then we have a bijection

$$\mathcal{H}(\mathbb{Q}) \backslash \mathcal{H}(\mathbb{A}) / S \xrightarrow{\sim} G, \qquad h \mapsto h(L \otimes \hat{\mathbb{Z}}) \cap V.$$

Let $\{h_j\}$ be a complete set of representatives of $\mathcal{H}(\mathbb{Q})\backslash\mathcal{H}(\mathbb{A})/S$ and let $\{L_j\}$ be the corresponding representatives of G under this bijection. Then

$$\int_{\mathcal{H}(\mathbb{Q})\backslash\mathcal{H}(\mathbb{A})} \theta(g,h;\varphi)\mathrm{d}h = \sum_j \int_{\mathcal{H}(\mathbb{Q})\backslash\mathcal{H}(\mathbb{Q})h_jS} \theta(g,h;\varphi)\mathrm{d}h.$$

Substituting h for hh_j and applying $\mathcal{H}(\mathbb{Q})\cap h_jSh_j^{-1} = \mathrm{Aut}(L_j)$, we obtain

$$\begin{aligned}\int_{\mathcal{H}(\mathbb{Q})\backslash\mathcal{H}(\mathbb{Q})h_jS} \theta\left(g,h;\varphi\right)\mathrm{d}h &= \int_{\mathcal{H}(\mathbb{Q})\backslash\mathcal{H}(\mathbb{Q})h_jSh_j^{-1}} \theta\left(g,hh_j;\varphi\right)\mathrm{d}h \\ &= \frac{1}{\#\,\mathrm{Aut}(L_j)}\int_{h_jSh_j^{-1}} \theta\left(g,hh_j;\varphi\right)\mathrm{d}h.\end{aligned}$$

Substituting h for $h_jhh_j^{-1}$, the third integral becomes

$$\begin{aligned}\int_S \theta\left(g,h_jh;\varphi\right)\mathrm{d}h &= \int_S \sum_{\underline{x}\in V^n} (\omega_\infty(g)\varphi_\infty\otimes\iota(e^{\underline{\mu}}))(h^{-1}h_j^{-1}\underline{x})\mathrm{d}h \\ &= \frac{\mathrm{vol}(S)}{|\,\mathrm{O}(D)|}\sum_{\sigma\in\mathrm{Iso}(D,L_j'/L_j)}\ \sum_{\underline{x}\in\sigma\underline{\mu}+L_j^n} (\omega_\infty(g)\varphi_\infty)(\underline{x}),\end{aligned}$$

where in the last equation we used that the canonical homomorphism $S\to\mathrm{O}(D)$ is surjective (see [55, Corollary 1.9.6.]). Combining these computations we find

$$\begin{aligned}&j(g,iI_n)^{m/2}\cdot\int_{\mathcal{H}(\mathbb{Q})\backslash\mathcal{H}(\mathbb{A})} \theta(g,h;\varphi)\mathrm{d}h \\ &= \frac{\mathrm{vol}(S)}{|\,\mathrm{O}(D)|}\sum_j \frac{1}{\#\,\mathrm{Aut}(L_j)}\sum_{\sigma\in\mathrm{Iso}(D,L_j'/L_j)}\ \sum_{\underline{x}\in\sigma\underline{\mu}+L_j^n} e^{\pi i\,\mathrm{tr}((\underline{x},\underline{x})Z)} \\ &= \frac{\mathrm{vol}(S)}{|\,\mathrm{O}(D)|}\sum_j \frac{1}{\#\,\mathrm{Aut}(L_j)}\sum_{\sigma\in\mathrm{Iso}(D,L_j'/L_j)} \langle\theta_{L_j}^{(n)}, e^{\sigma\underline{\mu}}\rangle \\ &= \frac{\mathrm{vol}(S)}{|\,\mathrm{O}(D)|}\sum_j \frac{1}{\#\,\mathrm{Aut}(L_j)}\sum_{\sigma\in\mathrm{Iso}(D,L_j'/L_j)} \langle\sigma^*\theta_{L_j}^{(n)}, e^{\underline{\mu}}\rangle.\end{aligned}$$

Finally, note that

$$\mathrm{vol}(S)\cdot\mu(G) = \mathrm{vol}(S)\cdot\sum_j\frac{1}{\#\,\mathrm{Aut}(L_j)} = \mathrm{vol}(\mathcal{H}(\mathbb{Q})\backslash\mathcal{H}(\mathbb{A})) = 1$$

by our choice of normalization. □

Chapter 2

Isotropic lifts

In this chapter we describe the isotropic lifts and prove a theorem that gives a precise statement about when modular forms for the Weil representation are induced from modular forms for smaller discriminant forms.

This chapter is based on the paper [52].

2.1 Isotropic subgroups and isotropic lifts

Let H be an isotropic subgroup of D. Then $H^\perp/H$ is a discriminant form of the same signature as D and order $|H^\perp/H| = |D|/|H|^2$. There is an *isotropic lift*

$$\uparrow_H := \uparrow_H^D \colon \mathbb{C}[(H^\perp/H)^n] \to \mathbb{C}[D^n]$$

defined by

$$\uparrow_H (e^{\underline{\gamma}+H^n}) = \sum_{\underline{\mu}\in H^n} e^{\underline{\gamma}+\underline{\mu}}$$

for $\underline{\gamma} \in (H^\perp)^n$ and an *isotropic descent*

$$\downarrow_H := \downarrow_H^D \colon \mathbb{C}[D^n] \to \mathbb{C}[(H^\perp/H)^n]$$

defined by

$$\downarrow_H (e^{\underline{\gamma}}) = \begin{cases} e^{\underline{\gamma}+H^n} & \text{if } \underline{\gamma} \in (H^\perp)^n, \\ 0 & \text{otherwise} \end{cases}$$

(cf. for example [11], [60] or [61]). The following results are easy to prove (see [53]).

Proposition 2.1.1. *Let D be a discriminant form of even signature and H an isotropic subgroup of D. Then the maps $\uparrow_H^D$ and $\downarrow_H^D$ are adjoint with respect to the* *inner products on* $\mathbb{C}[(H^\perp/H)^n]$ *and* $\mathbb{C}[D^n]$ *and commute with the Weil representations* $\rho^{(n)}_{H^\perp/H}$ *and* $\rho^{(n)}_D$. *In particular they map modular forms to modular forms. This implies that* $\uparrow^D_H$ *and* $\downarrow^D_H$ *as maps of modular forms are adjoint with respect to the Petersson scalar product.*

The isotropic lift is transitive in the following sense.

Proposition 2.1.2. *Let* D *be a discriminant form of even signature and* $H \subset K$ *isotropic subgroups of* D. *Then* $H \subset K \subset K^\perp \subset H^\perp$ *and* K/H *is an isotropic subgroup of* $H^\perp/H$ *with orthogonal complement* $K^\perp/H$. *There is a natural isomorphism* $(K^\perp/H)/(K/H) \cong K^\perp/K$. *Identifying the two groups we get*

$$\begin{aligned} \uparrow^D_H \circ \uparrow^{H^\perp/H}_{K/H} &= \uparrow^D_K \quad \textit{and} \\ \downarrow^{H^\perp/H}_{K/H} \circ \downarrow^D_H &= \downarrow^D_K . \end{aligned}$$

For an isotropic subgroup $H \subset D'$ and $\sigma \in \operatorname{Iso}(D, D')$ we easily check that $\tilde{\sigma}(\gamma + \sigma^{-1}H) := \sigma\gamma + H$ defines an element $\tilde{\sigma} \in \operatorname{Iso}((\sigma^{-1}H)^\perp/(\sigma^{-1}H), H^\perp/H)$. This implies that the diagram

$$\begin{array}{ccc} \mathbb{C}[D] & \xrightarrow{\quad\sigma_*\quad} & \mathbb{C}[D'] \\ \Big\downarrow{\scriptstyle \downarrow_{\sigma^{-1}H}} & & \Big\downarrow{\scriptstyle \downarrow_H} \\ \mathbb{C}[(\sigma^{-1}H)^\perp/(\sigma^{-1}H)] & \xrightarrow{\tilde{\sigma}_*} & \mathbb{C}[H^\perp/H] \end{array}$$

commutes.

Let $\mathcal{H}$ be the set of all non-trivial isotropic subgroups H of D. To simplify the notation we introduce the following subspaces of $\mathbb{C}[D]$:

$$\begin{aligned} \operatorname{im}(\uparrow) &= \sum_{H \in \mathcal{H}} \operatorname{im}(\uparrow_H) \quad \text{and} \\ \ker(\downarrow) &= \bigcap_{H \in \mathcal{H}} \ker(\downarrow_H). \end{aligned}$$

Because $\uparrow_H$ and $\downarrow_H$ are adjoint we find that

$$\begin{aligned} \operatorname{im}(\uparrow)^\perp &= \ker(\downarrow) \quad \text{and} \\ \operatorname{im}(\uparrow) &= \ker(\downarrow)^\perp \end{aligned} \tag{2.1.1}$$

with respect to the inner product on $\mathbb{C}[D]$.

2.2 Reduction to lifting pointwise

In this section we show that a modular form f is a linear combination of isotropically lifted modular forms if and only if for every τ on the upper half-plane $f(\tau)$ is a linear combination of lifts. In Theorem 2.2.4 we will argue further that for a discriminant form D all modular forms are linear combinations of isotropically lifted modular forms if and only if all of $\mathbb{C}[D]$ is in the image of the lifts.

Proposition 2.2.1. *Let $f \in \mathrm{M}_k(D)$ and let $V = \mathrm{span}_{\tau\in\mathbb{H}}(f(\tau)) \subset \mathbb{C}[D]$ and $(v_i)_{i\in I}$ an orthonormal basis of V with respect to $\langle\cdot,\cdot\rangle$. Then there exist $(f_i)_{i\in I}$ with $f_i \in \mathrm{M}_k(\mathrm{Mp}_2(N))$ such that*

$$f = \sum_{i\in I} f_i v_i.$$

In fact $f_i(\tau) = \langle f(\tau), v_i\rangle$.

Proof. Since for all $\tau \in \mathbb{H}$ the point $f(\tau)$ is in V and $(v_i)_{i\in I}$ is a orthonormal basis of V, the equality

$$\sum_{i\in I}\langle f(\tau), v_i\rangle v_i = f(\tau)$$

holds pointwise for all $\tau \in \mathbb{H}$. Hence, we only need to show that $f_i := \langle f, v_i\rangle \in \mathrm{M}_k(\mathrm{Mp}_2(N))$: For any $(M,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$ we have

$$\begin{aligned} f_i|_k[(M,\phi)] &= \langle f|_k[(M,\phi)], v_i\rangle \\ &= \langle \rho_D(M,\phi)f, v_i\rangle \\ &= \langle f, \rho_D(M,\phi)^{-1}v_i\rangle, \end{aligned} \tag{2.2.1}$$

so if $(M,\phi) \in \mathrm{Mp}_2(N)$, then $\rho_D(M,\phi)^{-1}$ acts trivially on v_i and f_i is invariant under $\mathrm{Mp}_2(N)$. The holomorphicity of f on $\mathbb{H}$ and at ∞ together with (2.2.1) implies holomorphicity of f_i on $\mathbb{H}$ and at the cusps. □

Now we can show that we can restrict to pointwise lifting.

Proposition 2.2.2. *Let D be a discriminant form and let $f \in \mathrm{M}_k(D)$. Then f is a linear combination of isotropically lifted modular forms if and only if $f(\tau) \in \mathrm{im}(\uparrow)$ for all $\tau \in \mathbb{H}$.*

Proof. If f is a linear combination of isotropically lifted modular forms, then it is clear that $f(\tau) \in \mathrm{im}(\uparrow)$. So let us assume that $f(\tau) \in \mathrm{im}(\uparrow)$ holds for all $\tau \in \mathbb{H}$. By Proposition 2.2.1 we can write

$$f = \sum_{i\in I} f_i v_i, \tag{2.2.2}$$

where $f_i \in \mathrm{M}_k(\mathrm{Mp}_2(N))$ and $(v_i)_{i\in I}$ is a basis of $V = \mathrm{span}_{\tau\in\mathbb{H}}(f(\tau))$. But by assumption $V \subset \mathrm{im}(\uparrow)$. So we can write

$$v_i = \sum_{H\in\mathcal{H}} \uparrow_H (v_{i,H}) \tag{2.2.3}$$

for suitable $v_{i,H} \in \mathbb{C}[H^\perp/H]$. Since f is a modular form for ρ_D, we have

$$f = \frac{1}{|\mathrm{Mp}_2(N)\backslash\mathrm{Mp}_2(\mathbb{Z})|} \sum_{(M,\phi)\in\mathrm{Mp}_2(N)\backslash\mathrm{Mp}_2(\mathbb{Z})} \rho_D(M,\phi)^{-1} f|_k[(M,\phi)].$$

For better readability we denote $A := |\mathrm{Mp}_2(N)\backslash\mathrm{Mp}_2(\mathbb{Z})|$. Applying (2.2.2) and (2.2.3), we get

$$\begin{aligned} f &= A^{-1}\cdot \sum_{(M,\phi)}\sum_{i\in I} f_i|_k(M,\phi)\rho_D(M,\phi)^{-1}v_i \\ &= A^{-1}\cdot \sum_{(M,\phi)}\sum_{i\in I} f_i|_k[(M,\phi)] \sum_{H\in\mathcal{H}} \rho_D(M,\phi)^{-1}\uparrow_H (v_{i,H}) \\ &= \sum_{H\in\mathcal{H}} \uparrow_H \left(A^{-1}\cdot \sum_{(M,\phi)}\sum_{i\in I} f_i|_k[(M,\phi)]\rho_{H^\perp/H}(M,\phi)^{-1}v_{i,H}\right), \end{aligned}$$

where

$$A^{-1}\cdot \sum_{(M,\phi)}\sum_{i\in I} f_i|_k(M,\phi)\rho_{H^\perp/H}(M,\phi)^{-1}v_{i,H} = \sum_{i\in I} F_{f_i,v_{i,H}}$$

is a modular form for $\rho_{H^\perp/H}$. □

We remark that [71, Theorem 115] is equivalent to Proposition 2.2.2, however our argument is shorter.

Clearly, if $\mathrm{im}(\uparrow) = \mathbb{C}[D]$, then all $f \in \mathrm{M}_k(D)$ are linear combinations of isotropic lifts. We want to argue that the other direction holds as well.

Proposition 2.2.3. *Let D be a discriminant form. Then*

$$\mathrm{span}\{f(\tau) \mid f \in \mathrm{M}_k(D),\ \tau\in\mathbb{H},\ k\in \tfrac{1}{2}\mathbb{Z}\} = \mathbb{C}[D].$$

Proof. Define $V := \mathrm{span}\{f(\tau) \mid f \in \mathrm{M}_k(D),\ \tau\in\mathbb{H},\ k\in \tfrac{1}{2}\mathbb{Z}\}$ and suppose that $V \subsetneq \mathbb{C}[D]$. Then $W := V^\perp$ is non-trivial. Since V is invariant under ρ_D and the Weil representation is a unitary action for the scalar product, also W is invariant under ρ_D. Therefore, (W,ρ_D) is a representation of $\mathrm{Mp}_2(\mathbb{Z})$ with $W \neq \{0\}$. For k large enough there exists a non-zero modular form f for (W,ρ_D) (This was shown in [42, section 3] for the integral weight case, the general case is treated in [16, Proposition 3.3]). But then also $f \in \mathrm{M}_k(D)$ and $f(\tau)\in W$ for all $\tau\in\mathbb{H}$, which contradicts the definition of W. □

We get

Theorem 2.2.4. *Let D be a discriminant form. Then all modular forms for the Weil representation are linear combinations of isotropically lifted modular forms if and only if* $\mathrm{im}(\uparrow) = \mathbb{C}[D]$.

Proof. By Proposition 2.2.2, $\mathrm{im}(\uparrow) = \mathbb{C}[D]$ implies that all modular forms for the Weil representation are linear combinations of isotropically lifted modular forms. If $\mathrm{im}(\uparrow) \subsetneq \mathbb{C}[D]$, then by Proposition 2.2.3 there exists some non-trivial modular form on $\mathrm{im}(\uparrow)^\perp$. By Proposition 2.2.2 this can not be a linear combination of isotropically lifted modular forms. □

We remark that the line of reasoning in this section can also be applied to other finite groups and representations (Recall that the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ descends to a representation of the finite group $\mathrm{Mp}_2(N)\backslash\mathrm{Mp}_2(\mathbb{Z})$).

2.3 The image of the lifts

In this section we want to investigate when $\mathrm{im}(\uparrow)$ contains all of $\mathbb{C}[D]$. We will show under what conditions we can write e^γ as a linear combination of lifts for any $\gamma \in D$. Since D and also any isotropic subgroup of D decomposes into its p-subgroups we can restrict ourselves to the case where the level of D is a power of a prime p. Therefore, in this section p is some fixed prime and D will always be a discriminant form of level a power of p.

The following lemma combines Lemmas 120 and 121 in [71].

Lemma 2.3.1. *Let $\gamma \in D$. Suppose that $\gamma^\perp$ contains an isotropic subgroup H isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$. Then $e^\gamma \in \mathrm{im}(\uparrow)$.*

Proof. The group H has $p^2 - 1$ elements of order p and therefore has $p + 1 = (p^2 - 1)/(p - 1)$ subgroups of order p. We denote them by $H_0, H_1, \ldots, H_p$. The inclusions $H_j \subset H \subset \langle\gamma\rangle^\perp$ imply $\langle\gamma\rangle \subset H^\perp \subset H_j^\perp$. Define

$$v = \sum_{j=1}^{p} \uparrow_{H_j}^{D} (e^{\gamma+H_j}) = \sum_{j=1}^{p} \sum_{\mu\in H_j} e^{\gamma+\mu} = pe^\gamma + \sum_{\mu\in H\backslash H_0} e^{\gamma+\mu}$$

and

$$w = \uparrow_{H_0}^{D} \Big(\sum_{\mu\in H_1\backslash\{0\}} e^{\gamma+\mu+H_0} \Big) = \sum_{\mu\in H_1\backslash\{0\}} \sum_{\beta\in H_0} e^{\gamma+\mu+\beta} = \sum_{\mu\in H\backslash H_0} e^{\gamma+\mu}$$

Then $e^\gamma = (v - w)/p$. □

A simple condition on D that ensures that the condition of Lemma 2.3.1 is satisfied for all $\gamma \in D$ is given in

Lemma 2.3.2. *If there exists an isotropic subgroup $H \subset D$ isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$ then for all $\gamma \in D$ the group $\gamma^\perp$ contains an isotropic subgroup $H \subset D$ isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$.*

Proof. Let $\gamma \in D$ and $H \subset D$ be an isotropic subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$. If $\gamma \in H^\perp$, take any subgroup of H isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$. If $\gamma \notin H^\perp$, we can choose generators α, β, μ of H such that $(\gamma, \mu) \neq 0 \bmod 1$. Since α, β and μ are of order p it follows that (α, γ), (β, γ) and (μ, γ) take values in $\frac{1}{p}\mathbb{Z}$. This implies that we can choose $x, y \in \mathbb{Z}$ such that

$$\begin{aligned}(\alpha + x\mu, \gamma) &= (\alpha, \gamma) + x(\mu, \gamma) = 0 \mod 1 \text{ and}\\ (\beta + y\mu, \gamma) &= (\beta, \gamma) + y(\mu, \gamma) = 0 \mod 1.\end{aligned}$$

Then $\langle \alpha + x\mu, \beta + y\mu \rangle$ is the desired group. □

Corollary 2.3.3. *If there exists an isotropic subgroup $H \subset D$ that is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$, then* $\operatorname{im}(\uparrow) = \mathbb{C}[D]$.

We will also need to understand when $e^\gamma \notin \operatorname{im}(\uparrow)$. Recall that $\mathcal{H}$ denotes the set of non-trivial isotropic subgroups. We define $\mathcal{H}' = \{H \in \mathcal{H} \mid |H| = p\}$ and

$$\begin{aligned}\operatorname{im}(\uparrow') &:= \sum_{H \in \mathcal{H}'} \operatorname{im}(\uparrow_H)\\ \ker(\downarrow') &:= \bigcap_{H \in \mathcal{H}'} \ker(\downarrow_H).\end{aligned}$$

Similar to equation (2.1.1) also $\operatorname{im}(\uparrow') = \ker(\downarrow')^\perp$. For the construction of e^γ as a linear combination of isotropic lifts in the proof of Lemma 2.3.1, only subgroups of order p are used. It turns out that this always suffices:

Lemma 2.3.4. *One has* $\operatorname{im}(\uparrow') = \operatorname{im}(\uparrow)$ *and* $\ker(\downarrow') = \ker(\downarrow)$.

Proof. First we show that $\ker(\downarrow') = \ker(\downarrow)$. The direction $\ker(\downarrow') \supset \ker(\downarrow)$ is clear, so let $v \in \ker(\downarrow')$ and let H be any non-trivial isotropic subgroup. Then H contains an isotropic subgroup of order p say H' and H/H' is an isotropic subgroup in $H'^\perp/H'$. Therefore

$$\downarrow_H^D(v) = \downarrow_{H/H'}^{H'^\perp/H'}(\downarrow_{H'}^D(v)) = \downarrow_{H/H'}^{H'^\perp/H'}(0) = 0$$

and $v \in \ker(\downarrow)$. Because $\operatorname{im}(\uparrow) = \ker(\downarrow)^\perp = \ker(\downarrow')^\perp = \operatorname{im}(\uparrow')$, we automatically get the other equality. □

We will later see that in some cases the condition in Lemma 2.3.1 is necessary for e^γ to be in $\mathrm{im}(\uparrow)$. For this we will need

Lemma 2.3.5. *Let $\gamma \in D$. Assume that $\gamma^\perp$ contains no isotropic subgroup $H \subset D$ isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$. Then for*

$$v := e^\gamma - \frac{1}{p-1} \sum_{\substack{H' \in \mathcal{H}' \\ H' \subset \gamma^\perp}} \sum_{\mu \in H' \setminus \{0\}} e^{\gamma+\mu}$$

we have $\downarrow_H (v) = 0$ for all non-trivial isotropic subgroups $H \subset \gamma^\perp$ and $\langle v, e^\gamma \rangle = 1$.

Proof. The assertion $\langle v, e^\gamma \rangle = 1$ is clear from the construction. Let H be any non-trivial isotropic subgroup with $H \subset \gamma^\perp$. Because of the assumption, H contains exactly one subgroup of order p say H'. The subgroup H' appears in the sum defining v. For any $K \in \mathcal{H}'$ with $K \subset \gamma^\perp$ and $\mu \in K \setminus \{0\}$ we get the following equivalence:

$$\gamma + \mu \in H^\perp \Leftrightarrow \mu \in H^\perp \Leftrightarrow \mu \in H' \Leftrightarrow K = H'$$

because otherwise $\langle \mu, H' \rangle$ would contradict the assumption of the lemma. So

$$\begin{aligned} \downarrow_H (v) &= \downarrow_H (e^\gamma) - \frac{1}{p-1} \sum_{\mu \in H' \setminus \{0\}} \downarrow_H (e^{\gamma+\mu}) \\ &= e^{\gamma+H} - \frac{1}{p-1} \sum_{\mu \in H' \setminus \{0\}} e^{\gamma+H} \\ &= 0. \end{aligned}$$

□

The case where p is an odd prime

In this subsection we assume that p is odd.

First we want to give a necessary condition for $e^\gamma \in \mathrm{im}(\uparrow)$:

Lemma 2.3.6. *Let $\gamma \in D$. If $e^\gamma \in \mathrm{im}(\uparrow)$ then $\gamma^\perp$ contains at least two isotropic subgroups of order p.*

Proof. We show the contraposition: Assume that $\langle \mu \rangle \subset \gamma^\perp$ is the only isotropic subgroup of order p in $\gamma^\perp$. Let $\mathrm{ord}(\gamma) = n$ and $\mathrm{q}(\gamma) = j/n$.
If $\gamma = 0$, it is clear because then $\gamma^\perp = D$, so assume $n > 1$. First assume that $n = p$ and $j = 0 \bmod p$. Then $\langle \gamma \rangle = \langle -\gamma \rangle = \langle \mu \rangle$ and so $\downarrow_{\langle \mu \rangle} (e^\gamma - e^{-\gamma}) = 0$.

Since for all other subgroups H of order p we have $\gamma, -\gamma \notin H^\perp$, by definition also $\downarrow_H (e^\gamma - e^{-\gamma}) = 0$. So $e^\gamma - e^{-\gamma} \in \ker(\downarrow)$. Since $\langle e^\gamma - e^{-\gamma}, e^\gamma \rangle = 1$,

$$e^\gamma \notin \ker(\downarrow)^\perp = \mathrm{im}(\uparrow).$$

Now we assume that $n > p$ or $(j, p) = 1$. We first want to show that $(\gamma + \mu)^\perp$ contains only one isotropic subgroup of order p as well, which then must also be $\langle \mu \rangle$:
If $j = 0 \bmod p$ and $n > p$ we have

$$\mathrm{q}(n/p \cdot \gamma) = n^2/p^2 \cdot j/n = \frac{nj}{p^2} = 0 \mod 1 \quad \text{and}$$
$$(n/p \cdot \gamma, \gamma) = 2n/p \cdot \mathrm{q}(\gamma) = 2n/p \cdot j/n = 0 \mod 1.$$

Hence, $\langle \mu \rangle = \langle n/p \cdot \gamma \rangle$ and so $\gamma + \mu = (1 + kn/p) \cdot \gamma$ for some $k = 1, \ldots, p-1$ and

$$((1 + kn/p) \cdot \gamma)^\perp = \gamma^\perp.$$

If $(j, p) = 1$, we can write

$$D = \langle \gamma \rangle \oplus \langle \gamma \rangle^\perp.$$

Note that $\mathrm{ord}(\gamma + \mu) = n$ and $\mathrm{q}(\gamma + \mu) = \mathrm{q}(\gamma)$, so we can also write

$$D = \langle \gamma + \mu \rangle \oplus \langle \gamma + \mu \rangle^\perp.$$

Since $\langle \gamma \rangle \cong \langle \gamma + \mu \rangle$, also $\langle \gamma \rangle^\perp \cong \langle \gamma + \mu \rangle^\perp$.
Now v from Lemma 2.3.5 is simply

$$v = e^\gamma - \frac{1}{p-1} \sum_{k=1}^{p-1} e^{\gamma + k\mu}$$

and $\langle \mu \rangle$ is the only isotropic subgroup of order p orthogonal to any component of v. So $\downarrow_{\langle \mu \rangle} (v) = 0$ implies $v \in \ker(\downarrow)$. Since $\langle v, e^\gamma \rangle = 1$,

$$e^\gamma \notin \ker(\downarrow)^\perp = \mathrm{im}(\uparrow).$$

□

Now we can give a condition that is equivalent to $e^\gamma \in \mathrm{im}(\uparrow)$ in many cases:

Proposition 2.3.7. *Let $\gamma \in D$ be of order n with $\mathrm{q}(\gamma) = j/n$. Assume that if $n > p$, then $j = 0 \bmod p$. Then $e^\gamma \in \mathrm{im}(\uparrow)$ if and only if $\gamma^\perp$ contains an isotropic subgroup $H \subset D$ isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$.*

Proof. If $\gamma^\perp$ contains an isotropic subgroup $H \subset D$ isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$ we can apply Lemma 2.3.1. So assume on the other hand that $\gamma^\perp$ contains no isotropic subgroup $H \subset D$ isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$. We construct a $v \in \ker(\downarrow)$ with $\langle v, e^\gamma \rangle = 1$. Then

$$e^\gamma \notin \ker(\downarrow)^\perp = \mathrm{im}(\uparrow).$$

If $\gamma = 0$, then $\gamma^\perp = D$ and Lemma 2.3.5 yields v. So assume that $n > 1$. First consider the case $j = 0 \bmod p$: as in the previous lemma $n/p \cdot \gamma$ is isotropic and orthogonal to γ. And for any isotropic subgroup H with $\gamma \in H^\perp$, also $n/p \cdot \gamma \in H^\perp$. Hence, $\langle n/p \cdot \gamma \rangle$ must be the only isotropic subgroup of order p in $\gamma^\perp$, because if $\gamma^\perp$ contained any other isotropic subgroup of order p, say H, then $\langle H, n/p \cdot \gamma \rangle$ would contradict the assumption on γ. But then by Lemma 2.3.6 $e^\gamma \notin \mathrm{im}(\uparrow)$.
Now assume $(j, p) = 1$ and so $n = p$ by the assumption of the proposition. Since $(\gamma, \beta) = 0$ is equivalent to $(-\gamma, \beta) = 0$, also $-\gamma$ satisfies our assumption. Let v_1 and v_2 be the elements from Lemma 2.3.5 for γ and $-\gamma$ respectively. We define

$$v := v_1 - v_2.$$

For any non-trivial isotropic subgroup $H \subset \gamma^\perp$ we have

$$\downarrow_H (v) = \downarrow_H (v_1) - \downarrow_H (v_2) = 0 - 0 = 0.$$

Now we want to show that also $\downarrow_H (v) = 0$ when $H \not\subset \gamma^\perp$. By Lemma 2.3.4 it suffices to consider isotropic subgroups of order p. So let μ be any isotropic element of order p with $(\gamma, \mu) \neq 0 \bmod 1$. We show that whenever $(\gamma + \mu_1, \mu) = 0 \bmod 1$ for some isotropic $\mu_1 \in \gamma^\perp$ with $\mathrm{ord}(\mu_1) = p$, then there exists exactly one isotropic $\mu_2 \in \gamma^\perp$ with $\mathrm{ord}(\mu_2) = p$ such that

$$\gamma + \mu_1 = -\gamma + \mu_2 \mod \langle \mu \rangle, \text{ i.e.}$$
$$\gamma + \mu_1 + l \cdot \mu = -\gamma + \mu_2$$

for some $l = 0, \dots, p-1$. We can reverse the roles of γ and $-\gamma$. This shows that the terms in $\downarrow_{\langle \mu \rangle} (v_1)$ and $\downarrow_{\langle \mu \rangle} (v_2)$ cancel each other. So assume that $(\gamma, \mu) \neq 0 \bmod 1$, but $(\gamma + \mu_1, \mu) = 0 \bmod 1$. We need to find a suitable l such that

$$\mu_2 := 2\gamma + \mu_1 + l \cdot \mu$$

is isotropic, orthogonal to γ and of order p. We have

$$\begin{aligned} \mathrm{q}(\mu_2) &= \mathrm{q}(2\gamma + \mu_1 + l \cdot \mu) \\ &= 4\,\mathrm{q}(\gamma) + 2l(\gamma, \mu) + l(\mu_1, \mu) \\ &= 4\,\mathrm{q}(\gamma) + l(\gamma + \mu_1, \mu) + l(\gamma, \mu) \\ &= 4j/p + l(\gamma, \mu). \end{aligned}$$

Since $(\gamma, \mu) \neq 0 \bmod 1$, there exists exactly one $l \bmod p$ such that μ_2 is isotropic. With said l we have

$$\begin{aligned}(\gamma, \mu_2) &= (\gamma, 2\gamma + \mu_1 + l \cdot \mu) \\ &= 4\,\mathrm{q}(\gamma) + (\gamma, \mu_1) + l(\gamma, \mu) \\ &= 4j/p + l(\gamma, \mu) = 0 \mod 1.\end{aligned}$$

Clearly $p \cdot \mu_2 = 0$. Furthermore, μ_2 cannot be 0, because then $l \cdot \mu = -(2\gamma + \mu_1)$ and

$$0 = \mathrm{q}(l\mu) = 4\,\mathrm{q}(\gamma) = 4j/p \mod 1,$$

which is a contradiction. □

We will later see an example of a γ of order $n > p$ and norm j/p with $(j, p) = 1$ where $\gamma^\perp$ contains no isotropic subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$ but still $e^\gamma \in \mathrm{im}(\uparrow)$. Since for components of level higher than p, elements of order p are always isotropic, it is useful to decompose $D = A \oplus B$ where A denotes the sum over the irreducible components of order p and B the sum over the remaining components. So $A = p^{\epsilon n}$ for some $\epsilon = \pm 1$ and $n \geq 0$. To better understand the isotropic subgroups of $p^{\pm n}$ we need

Proposition 2.3.8. *Let $D = p^{\epsilon n}$ and let $H \subset D$ be a maximal isotropic subgroup of D. Then $H \cong (\mathbb{Z}/p\mathbb{Z})^r$ with*

$$r = \begin{cases} n/2 & \text{if } n \text{ is even and } \epsilon = \left(\frac{-1}{p}\right)^{n/2} \\ (n-1)/2 & \text{if } n \text{ is odd} \\ (n-2)/2 & \text{if } n \text{ is even and } \epsilon = -\left(\frac{-1}{p}\right)^{n/2} \end{cases}.$$

Proof. Since H is maximal, the discriminant form $H^\perp/H$ contains no non-trivial isotropic elements. It is well-known and not difficult to prove that the only p-adic discriminant forms with this property are 0, $p^{\pm 1}$ and $p^{-\varepsilon 2}$ with $\varepsilon = \left(\frac{-1}{p}\right)$. Because $|H^\perp/H| = |D|/|H|^2$, we find that

$$\mathrm{rk}(H^\perp/H) = n - 2r.$$

For n odd, this proves the claim and for even n it follows from $\mathrm{sign}(H^\perp/H) = \mathrm{sign}(D) \bmod 8$. □

Now we want to see for which discriminant forms $e^\gamma \in \mathrm{im}(\uparrow)$ for all γ. First we find those discriminant forms that satisfy the general condition of Corollary 2.3.3

Proposition 2.3.9. *Assume that D contains no isotropic subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$. Then D satisfies one of the following conditions:*

(i) D has rank two or less.

(ii) D has rank three and at least one Jordan component is of level p.

(iii) D has rank four and $D = p^{-\epsilon 2} q_1^{\pm 1} q_2^{\pm 1}$, where $\epsilon = \left(\frac{-1}{p}\right)$ and q_1, q_2 are powers of p and can also be p.

(iv) D has rank five and $D = p^{-4} q^{\pm 1}$, where q is a power of p and can also be p.

(v) D has rank six and $D = p^{-\epsilon 6}$, where $\epsilon = \left(\frac{-1}{p}\right)$.

Proof. We choose a Jordan decomposition of D and write $D = A \oplus B$ where A denotes the sum over the irreducible components of order p and B the sum over the remaining components. Since all elements of order p in B are isotropic and orthogonal to each other, B must have rank less than three. If it has rank two, $p^{\pm n}$ cannot contain any non-trivial isotropic element, so it is equal to one of

$$0,\ p^{\pm 1},\ p^{-\left(\frac{-1}{p}\right)2}.$$

If B has rank one, $p^{\pm n}$ cannot contain any isotropic subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$, so it is equal to one of

$$0,\ p^{\pm 1},\ p^{\pm 2},\ p^{\pm 3},\ p^{-4}.$$

If B is trivial, $p^{\pm n}$ cannot contain any isotropic subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$, so it is equal to one of

$$0,\ p^{\pm 1},\ p^{\pm 2},\ p^{\pm 3},\ p^{\pm 4},\ p^{\pm 5},\ p^{-\left(\frac{-1}{p}\right)6}.$$

It is easy to check that the discriminant forms in this list are exactly the ones obtained from the conditions stated in the proposition. □

Finally, we check for which discriminant forms appearing in Proposition 2.3.9 indeed $\mathrm{im}(\uparrow) = \mathbb{C}[D]$.

Theorem 2.3.10. *Let D be a discriminant form of level a power of an odd prime p. Then $\mathrm{im}(\uparrow) = \mathbb{C}[D]$ unless D is one of the following:*

(i) D has rank two or less.

(ii) D has rank three and at least one Jordan component is of level p.

(iii) D has rank four and $D = p^{-\epsilon 2} q_1^{\pm 1} q_2^{\pm 1}$*, where* $\epsilon = \left(\frac{-1}{p}\right)$ *and* q_1, q_2 *are powers of* p *and can also be* p*.*

(iv) D has rank five and is of level p*.*

Proof. If D contains an isotropic subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$, then by Corollary 2.3.3 $\mathrm{im}(\uparrow) = \mathbb{C}[D]$. Therefore, we consider the remaining discriminant forms, described in Proposition 2.3.9. We show that in all cases except $p^{-\left(\frac{-1}{p}\right)6}$ and $p^{-4}q^{\pm 1}$, there is an element $e^\gamma \notin \mathrm{im}(\uparrow)$. For $p^{-\left(\frac{-1}{p}\right)6}$ and $p^{-4}q^{\pm 1}$, we show that $\mathrm{im}(\uparrow) = \mathbb{C}[D]$. We begin with the latter two cases.
So let $\gamma \in p^{-\left(\frac{-1}{p}\right)6}$. We first assume that $\gamma \neq 0$. If $\mathrm{q}(\gamma) = 0 \mod 1$, we can write

$$D = \langle \gamma, \mu \rangle \oplus p^{-4},$$

for some $\mu \in D$ where $\langle \gamma, \mu \rangle \cong p^{\left(\frac{-1}{p}\right)2}$. The component p^{-4} contains a non-trivial isotropic element, say β. Then $\langle \gamma, \beta \rangle$ is an isotropic subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$ in $\gamma^\perp$ and we can apply Lemma 2.3.1. Now assume that $\mathrm{q}(\gamma) = x/p$ with $\left(\frac{2x}{p}\right) = \epsilon = \pm 1$. Then we can write

$$D = \langle \gamma \rangle \oplus p^{+4} \oplus p^{-\epsilon\left(\frac{-1}{p}\right)}.$$

The component p^{+4} contains an isotropic subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$ and we can again apply Lemma 2.3.1. We have seen that $p^{-\left(\frac{-1}{p}\right)6}$ contains isotropic subgroups H isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$. Obviously $H \subset 0^\perp = D$, hence, also $e^0 \in \mathrm{im}(\uparrow)$.
Now let $D = p^{-4}q^{\pm 1}$ with $q > p$ and write

$$D = p^{-4} \oplus \langle \gamma \rangle.$$

Let $\mathrm{q}(\gamma) = j/q$ with $(j, p) = 1$. It is not difficult to see that for any $\beta \in p^{-4}$, there exists a non-trivial isotropic $\mu \in p^{-4} \cap \beta^\perp$. And so for any element of the form $\beta + x\gamma$ with $x = 0 \mod p$, the isotropic group $H = \langle \mu, q/p \cdot \gamma \rangle$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$ and in $(\beta + x\gamma)^\perp$ because

$$(q/p \cdot \gamma, \beta + x\gamma) = 2xq/p \cdot \mathrm{q}(\gamma) = 0 \mod 1.$$

If $(x, p) = 1$, we can w.l.o.g. assume that $\beta = 0$, because one can also decompose $D = p^{-4} \oplus \langle x\gamma + \beta \rangle$. Let I_p be the set of isotropic elements in p^{-4} of order p. We define

$$
\begin{aligned}
v &:= \sum_{\mu \in I_p} \uparrow_{\langle \mu \rangle} (e^{\gamma + \langle \mu \rangle}) \\
w &:= \sum_{\substack{\mu, \beta \in I_p \\ (\mu,\beta) = -2j/p}} \uparrow_{\langle (q/p)\cdot\gamma + \beta \rangle} (e^{\gamma + \mu + \langle (q/p)\cdot \gamma + \beta \rangle}) \\
u_l &:= \sum_{\substack{\mu \in I_p, \beta \in p^{-4} \\ (\mu,\beta) = 0 \\ \mathrm{q}(\beta) = -2lj/p}} \uparrow_{\langle \mu \rangle} (e^{(1+lq/p)\gamma + \beta + \langle \mu \rangle})
\end{aligned}
$$

for $l = 1, \ldots, p-1$. Then we have

$$
v = |I_p| \cdot e^{\gamma} + (p-1) \sum_{\mu \in I_p} e^{\gamma + \mu}.
$$

The terms in w are of the form $e^{(1+lq/p)\gamma+\alpha}$, where $\alpha = \mu + l\beta$ with $\mathrm{q}(\alpha) = l(\mu,\beta) = -2lj/p$ and $(\alpha,\mu) = -2lj/p$. If we on the other hand start with an $l = 1, \ldots, p-1$ and $\alpha \in p^{-4} \setminus \{0\}$ with $\mathrm{q}(\alpha) = -2lj/p$, then we can find a $\mu \in I_p$ with $(\alpha, \mu) = -2lj/p$. Furthermore, $\beta := l^{-1}(\alpha - \mu) \in I_p$ and satisfies $(\mu, \beta) = -2j/p$ and $\alpha = \mu + l\beta$. We denote the number of such μ for a given α by a_l. This number does not depend on the choice of α because all such α are in the same orbit under the automorphism group of p^{-4}. For $l = 0$ let us denote the number of $\beta \in I_p$ with $(\beta, \mu) = -2j/p \bmod 1$ for a given $\mu \in I_p$ by $a_0 > 0$, which again does not depend on the choice of μ. Then we find

$$
w = \sum_{l=0}^{p-1} a_l \sum_{\substack{\alpha \in p^{-4} \setminus \{0\} \\ \mathrm{q}(\alpha) = -2lj/p}} e^{(1+lq/p)\gamma + \alpha}.
$$

The terms in u_l are of the form $e^{(1+lq/p)\gamma+\alpha}$, where $\alpha = m\mu + \beta$ with $\mathrm{q}(\alpha) = \mathrm{q}(\beta) = -2lj/p$ and $(\alpha, \mu) = 0$. Again for any given $l = 1, \ldots, p-1$ and $\alpha \in p^{-4} \setminus \{0\}$ with $\mathrm{q}(\alpha) = -2lj/p$ we find a $\mu \in I_p$ with $(\alpha, \mu) = 0$. Then for every $m = 0, \ldots, p-1$ we find that $\beta := \alpha - m\mu \in p^{-4}$ satisfies $(\mu, \beta) = 0$, $\mathrm{q}(\beta) = -2lj/p$ and $\alpha = m\mu + \beta$. We denote the number of such μ for a given α by $b_l > 0$, which again does not depend on the choice of α. Then we find

$$
u_l = p b_l \sum_{\substack{\alpha \in p^{-4} \setminus \{0\} \\ \mathrm{q}(\alpha) = -2lj/p}} e^{(1+lq/p)\gamma + \alpha}.
$$

Together we get

$$
\frac{1}{|I_p|} \left[v - \frac{p-1}{a_0} \cdot w + \sum_{l=1}^{p-1} \frac{(p-1)a_l}{a_0 p b_l} \cdot u_l \right] = e^{\gamma}.
$$

Now we go through the four conditions provided in the theorem and show that in each case we can always find a γ for which e^γ is not in $\mathrm{im}(\uparrow)$:
If D has rank two or less, we can write $D = \langle\beta\rangle \oplus \langle\gamma\rangle$, where β and γ are either anisotropic or trivial. So $\gamma^\perp = \langle\beta\rangle$ contains at most one isotropic subgroup of order p. By Lemma 2.3.6 $e^\gamma \notin \mathrm{im}(\uparrow)$.
Now let $D = \langle\mu\rangle \oplus \langle\beta\rangle \oplus \langle\gamma\rangle$, where μ, β and γ are all anisotropic and $\mathrm{q}(\mu) = x/p \bmod 1$ for some integer x coprime to p. Then $\gamma^\perp = \langle\mu\rangle \oplus \langle\beta\rangle$ and $\beta^\perp = \langle\mu\rangle \oplus \langle\gamma\rangle$. If $\mathrm{ord}(\beta) = n > p$, then $\langle n/p \cdot \beta\rangle$ is the only isotropic subgroup of order p in $\gamma^\perp$ and so $e^\gamma \notin \mathrm{im}(\uparrow)$. If $\mathrm{ord}(\beta) = p$, the only subgroup in $\beta^\perp$ isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$ is $\langle\mu\rangle \oplus \langle\mathrm{ord}(\gamma)/p \cdot \gamma\rangle$, which is not isotropic since μ is not. So by Proposition 2.3.7 $e^\beta \notin \mathrm{im}(\uparrow)$.
Now let $D = p^{-\left(\frac{-1}{p}\right)2} \oplus \langle\beta\rangle \oplus \langle\gamma\rangle$, where β and γ are anisotropic. Similar to before, if β is of order $n > p$, then $\langle n/p \cdot \beta\rangle$ is the only isotropic subgroup of order p in $\gamma^\perp$ and we can apply Lemma 2.3.6. If $\mathrm{ord}(\beta) = p$, then we can assume that γ has order p as well, because otherwise we are in the same situation as before just with the roles of β and γ reversed. And so $\beta^\perp \cong p^{\pm 3}$, which contains no isotropic subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$ and so by Proposition 2.3.7 $e^\beta \notin \mathrm{im}(\uparrow)$.
Finally, let $D = p^{-4} \oplus \langle\gamma\rangle$, with γ of order p. Then $\gamma^\perp = p^{-4}$, which contains no isotropic subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$ and so again by Proposition 2.3.7 $e^\gamma \notin \mathrm{im}(\uparrow)$. □

The case where $p = 2$

Now we consider the prime $p = 2$. The situation is similar to p odd, but more complicated.
We assume that D is a discriminant form of level a power of 2. We define a graph $G_D = (V, E)$ with set of vertices V and set of edges E by

$$\begin{aligned} V &:= D, \\ E &:= \{\{\gamma, \beta\} \subset V \mid \mu := \gamma - \beta, \mathrm{ord}(\mu) = 2, \\ &\qquad \mathrm{q}(\mu) = (\mu, \gamma) = (\mu, \beta) = 0 \bmod 1\}, \end{aligned}$$

i.e. there is an edge between γ and β if and only if $\{\gamma, \beta\}$ is a coset in $H^\perp/H$ for some isotropic subgroup H of order 2. If for $\gamma_1, \ldots, \gamma_n \in V$ there is an edge between γ_i and γ_{i+1} for $i = 1, \ldots, n-1$ and an edge between γ_n and γ_1, we call $(\gamma_1, \ldots, \gamma_n)$ an *isotropic cycle* of length n. Recall that a graph $G = (V, E)$ is called *bipartite* if one can partition V into two disjoint sets A and B such that for no $e \in E$ we have $e \subset A$ or $e \subset B$. We have

Proposition 2.3.11. *Let $\gamma \in D$. Then the following are equivalent:*

(i) $e^\gamma \in \mathrm{im}(\uparrow)$

(ii) The connected component of G_D containing γ is not bipartite

(iii) G_D contains an isotropic cycle containing γ that has odd length

Proof. We first show (i) implies (ii) by contraposition. Suppose that the connected component $G' = (V', E')$ of G_D containing γ was bipartite with decomposition $V' = A \cup B$. We can assume that $\gamma \in A$ and define

$$v := \sum_{\beta \in A} e^\beta - \sum_{\beta \in B} e^\beta.$$

Now let H be any isotropic subgroup of order 2. If $\beta \in H^\perp$ for some $\beta \in A$, then $\beta + H$ is an edge in G' and so the other element in this coset is in B. By the same reasoning if $\beta \in B$ the other element of $\beta + H$ is in A. Therefore

$$\begin{aligned} \downarrow_H (v) &= \sum_{\beta \in A} \downarrow_H (e^\beta) - \sum_{\beta \in B} \downarrow_H (e^\beta) \\ &= \sum_{\beta \in A} \downarrow_H (e^\beta) - \sum_{\beta \in A} \downarrow_H (e^\beta) \\ &= 0. \end{aligned}$$

By Lemma 2.3.4 this implies that $v \in \ker(\downarrow)$. Since $\langle v, e^\gamma \rangle = 1$, we know

$$e^\gamma \notin \ker(\downarrow)^\perp = \mathrm{im}(\uparrow).$$

Now assume that the connected component $G' = (V', E')$ of G_D containing γ is not bipartite. We want to show that G' contains an isotropic cycle containing γ that has odd length. It is a well-known fact from graph theory that a graph is bipartite if and only if it contains no cycle of odd length. So let $(\beta_1, \ldots, \beta_n)$ with n odd be an isotropic cycle in G', which must exist by assumption. Since G' is connected there exists a path $(\gamma, \gamma_1, \ldots, \gamma_m, \beta_1)$ in G'. But then

$$(\gamma, \gamma_1, \ldots, \gamma_m, \beta_1, \ldots, \beta_n, \beta_1, \gamma_m, \ldots, \gamma_1)$$

is an isotropic cycle containing γ of length $1 + m + n + 1 + m = 2m + 2 + n$, which is odd.

Finally, we want to show (iii) implies (i) so let $(\gamma_1, \ldots, \gamma_n)$ be an isotropic cycle with n odd and $\gamma_1 = \gamma$ and set $\gamma_{n+1} = \gamma_1$. Let $\mu_i = \gamma_{i+1} - \gamma_i$ for $i = 1, \dots, n$. By definition μ_i is isotropic and orthogonal to γ_i, so

$$-\frac{1}{2}\sum_{i=1}^{n}(-1)^i \uparrow_{\langle\mu_i\rangle} (e^{\gamma_i})$$

is well-defined and in $\mathrm{im}(\uparrow)$. We have

$$\begin{aligned}-\frac{1}{2}\sum_{i=1}^{n}(-1)^i \uparrow_{\langle\mu_i\rangle} (e^{\gamma_i}) &= -\frac{1}{2}\sum_{i=1}^{n}(-1)^i(e^{\gamma_i} + e^{\gamma_i+\mu_i}) \\ &= -\frac{1}{2}\sum_{i=1}^{n}(-1)^i(e^{\gamma_i} + e^{\gamma_{i+1}}) \\ &= \frac{1}{2}(e^{\gamma_1} + e^{\gamma_{n+1}}) = e^{\gamma}.\end{aligned}$$

□

If $\gamma^\perp$ contains an isotropic subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ then Lemma 2.3.1 implies that in this case an isotropic cycle of odd length containing γ exists. In fact if

$$\{0, \mu_1, \mu_2, \mu_1 + \mu_2\}$$

is such a group, then

$$(\gamma, \gamma + \mu_1, \gamma + \mu_1 + \mu_2)$$

is an isotropic cycle containing γ of length 3.
Lemma 2.3.6 also holds for $p = 2$:

Lemma 2.3.12. *Let $\gamma \in D$. If $e^\gamma \in \mathrm{im}(\uparrow)$ then $\gamma^\perp$ contains at least two isotropic subgroups of order* 2.

Proof. Suppose that μ_1 is the only isotropic element of order 2 in $\gamma^\perp$ and that $(\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_{n-1})$ is an isotropic cycle of length n with n odd and $\gamma_0 = \gamma$. Then $\mu_1 = \gamma_1 - \gamma_0$ and we set $\mu_i = \gamma_i - \gamma_{i-1}$ for $i = 2, \dots, n-1$ and $\mu_n = \gamma_0 - \gamma_{n-1}$. Note that $\gamma_i = \gamma + \sum_{j=1}^{i} \mu_j$, $(\mu_i, \gamma_{i-1}) = (\mu_i, \gamma_i) = 0 \bmod 1$ and $\mathrm{q}(\gamma_i) = \mathrm{q}(\gamma) \bmod 1$ for all $i = 1, \dots n$. If $n = 3$, then $\{0, \mu_1, \mu_2, \mu_3\}$ is an isotropic subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ in $\gamma^\perp$. Hence, $n \geq 5$. We show that we can construct an isotropic cycle of length $n-2$. Then, recursively we can find an isotropic cycle of length 3, which is a contradiction. If $\mu_i = \mu_{i+1}$ for some $i = 1, \dots, n$, then $(\gamma_0, \dots, \gamma_{i-1}, \gamma_{i+2}, \dots, \gamma_n)$ is an isotropic cycle of length $n-2$. So assume that $\mu_i \neq \mu_{i+1}$, in particular $\mu_1 \neq \mu_2$. This implies

$$(\mu_2, \mu_1) = (\mu_2, \gamma) = 1/2 \mod 1.$$

If $\mu_3 = \mu_1$, then $\gamma_3 = \gamma + \mu_2$ and so

$$(\gamma, \mu_2) = \mathrm{q}(\gamma_3) - \mathrm{q}(\gamma) - \mathrm{q}(\mu_2) = 0 \mod 1,$$

which is a contradiction. Therefore, $(\mu_3, \gamma) = 1/2 \bmod 1$ and so $(\mu_3, \mu_1 + \mu_2) = 1/2 \bmod 1$. This implies that $\mu_1 + \mu_2 + \mu_3$ is isotropic and in $\gamma^\perp$, i.e. equal to μ_1. But then $(\gamma_0, \gamma_3, \ldots, \gamma_{n-1})$ is an isotropic cycle of length $n - 2$. □

As we did for odd primes we want to find those discriminant forms that do not contain an isotropic subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. Again it is useful to decompose the discriminant form D into those components where all elements of order 2 are isotropic, and those where this is not the case. The former is the case for Jordan components of type $4_{II}^{\pm n}$ and irreducible components generated by elements of order divided by 8. The latter therefore consists of components of type $2_{II}^{\pm n}$, $2_t^{\pm n}$ and $4_s^{\pm m}$. Furthermore note that

$$2_t^\epsilon \cong 2_{t+4}^{-\epsilon}$$

and $\epsilon e(t/8)$ and $\epsilon \left(\frac{t}{2}\right)$ are invariant under this change of symbols. Therefore, we can always assume that $\epsilon = +1$ and $2_t^{\epsilon n}$ is generated by pairwise orthogonal elements $\mu_1, \ldots, \mu_n$ with $\mathrm{q}(\mu_i) = t_i/4 \bmod 1$, where $t_i = \pm 1$.

For discriminant forms of level p odd it was easy to see, when they do not contain any isotropic subgroups of large rank. For discriminant forms consisting of components of type $2_{II}^{\pm n}$, $2_t^{\pm n}$ and $4_s^{\pm m}$ it is more difficult. We will do this in the next three lemmata.

Lemma 2.3.13. *Let $\mathcal{D}_1'$ be the set of discriminant forms D such that D is a sum of components of type $2_{II}^{\pm n}$, $2_t^{\pm n}$ and $4_s^{\pm m}$ and D contains no non-trivial isotropic elements. Then $\mathcal{D}_1'$ consists of the following discriminant forms:*

$$\begin{aligned}
&0,\ 2_{II}^{-2},\ 2_t^{\pm 1},\\
&2_t^{\pm 2},\ \textit{where } t = 2 \bmod 4,\\
&2_t^{\epsilon 3},\ \textit{where } \epsilon \left(\frac{t}{2}\right) = -1,\\
&4_t^{\pm 1},\ 2_t^{\pm 1} 4_s^{\pm 1}.
\end{aligned}$$

Proof. Let D be a sum of components of type $2_{II}^{\pm n}$, $2_t^{\pm n}$ and $4_s^{\pm m}$. First we assume that D is of type $2_{II}^{\pm n}$. By definition 2_{II}^{+2} contains a non-trivial isotropic element and if $n \geq 4$, we can write $2_{II}^{\pm n} \cong 2_{II}^{+2} \oplus 2_{II}^{\pm(n-2)}$, so that only 0 and 2_{II}^{-2} are in $\mathcal{D}_1'$.

Now we assume that D has level 4, i.e. D is of type $2_t^{\pm n}$. It follows from Proposition 1.1.3 that $2_t^{\pm 1}$, $2_t^{\pm 2}$ with $t = 2 \bmod 4$ and $2_t^{\epsilon 3}$ with $\epsilon\left(\frac{t}{2}\right) = -1$ are the only discriminant forms that do not contain non-trivial isotropic elements.
Now we assume that D has level 8, so D is of the form $4_s^{\pm n}$, $2_{II}^{-2}4_s^{\pm n}$ or $2_t^{\pm m}4_s^{\pm n}$. If $n \geq 2$ then it is again easy to see that there is an isotropic element of order 2. So $n = 1$ and $4_s^{\pm 1}$ is generated by an element γ and $\mathrm{q}(2\gamma) = 1/2 \bmod 1$. Therefore, the other component must not contain a non-trivial element of norm 0 or 1/2. This is the case only for 0 and $2_t^{\pm 1}$. □

Now we consider isotropic subgroups isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.

Lemma 2.3.14. *Let $\mathcal{D}_2'$ be the set of discriminant forms D such that D is a sum of components of type $2_{II}^{\pm n}$, $2_t^{\pm n}$ and $4_s^{\pm m}$ and D contains no isotropic subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Then $\mathcal{D}_2'$ consists of the following discriminant forms:*

$$\begin{aligned}
&0,\ 2_{II}^{\pm 2},\ 2_{II}^{-4},\\
&2_t^{\pm n},\ \textit{where } n \leq 3\\
&2_t^{\epsilon 4},\ \textit{where } \epsilon e(t/8) \neq 1\\
&2_t^{\epsilon 5},\ \textit{where } \epsilon\left(\frac{t}{2}\right) = -1\\
&4_s^{\pm 1},\ 2_{II}^{\pm 2}4_s^{\pm 1},\\
&2_t^{\pm n}4_s^{\pm 1},\ \textit{where } n \leq 3\\
&4_s^{\pm 2},\ 2_t^{\pm 1}4_s^{\pm 2}.
\end{aligned}$$

Proof. Let D be a sum of components of type $2_{II}^{\pm n}$, $2_t^{\pm n}$ and $4_s^{\pm m}$. First we assume that D is of type $2_{II}^{\pm n}$. Of course $2_{II}^{\pm 2} \cong (\mathbb{Z}/2\mathbb{Z})^2$, which contains an anisotropic element and so $2_{II}^{\pm 2} \in \mathcal{D}_2'$. If $n \geq 4$ we can write $2_{II}^{\pm n} \cong 2_{II}^{+2} \oplus 2_{II}^{\pm(n-2)}$ and $2_{II}^{\pm(n-2)}$ must be in $\mathcal{D}_1'$, so we only get $2_{II}^{-4} \in \mathcal{D}_2'$.
Now we assume that D has level 4. Then D is of type $2_t^{\epsilon n}$. If $D \in \mathcal{D}_1'$ then also $D \in \mathcal{D}_2'$. Otherwise D contains an isotropic element γ of order 2 and $\langle\gamma\rangle^\perp/\langle\gamma\rangle \cong 2_t^{\epsilon(n-2)}$ or $\langle\gamma\rangle^\perp/\langle\gamma\rangle \cong 2_{II}^{\epsilon'(n-2)}$, with $\epsilon' = \epsilon(-1)^{(t-4)/2}$. Now $D \in \mathcal{D}_2'$ if and only if $\langle\gamma\rangle^\perp/\langle\gamma\rangle \in \mathcal{D}_1'$. This shows that $D \in \mathcal{D}_2'$ when $n \leq 3$. For $n = 4$ we see that $D \in \mathcal{D}_2$ if $\epsilon e(t/8) = e(\mathrm{sign}(D)/8) \neq 1$ and for $n = 5$ if $\epsilon\left(\frac{t}{2}\right) = -1$.
Now we assume that D has level 8 and so $D = 2_{II}^{\pm m}4_s^{\pm n}$ or $D = 2_t^{\pm m}4_s^{\pm n}$. If $n = 1$ then $4_s^{\pm 1}$ is generated by an element γ of order 4 and $\mathrm{q}(2\gamma) = 1/2 \bmod 1$. So $2_{II}^{\pm m}$ respectively $2_t^{\pm m}$ must not contain any isotropic subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$, but also no distinct μ_1, μ_2 with $(\mu_1, \mu_2) = 0 \bmod 1$ and $\mathrm{q}(\mu_1) = \mathrm{q}(\mu_2) = 1/2 \bmod 1$, because in the latter case $\mu_1 + 2\gamma$ and $\mu_2 + 2\gamma$ generate an isotropic subgroup

isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. We have already seen that only

$$\begin{aligned} &0,\ 2_{II}^{\pm 2},\ 2_{II}^{-4}, \\ &2_t^{\pm n},\ \text{where } n \le 3 \\ &2_t^{\epsilon 4},\ \text{where } \epsilon e(t/8) \neq 1 \\ &2_t^{\epsilon 5},\ \text{where } \epsilon \left(\frac{t}{2}\right) = -1 \end{aligned}$$

contain no isotropic subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Going through this finite list of discriminant forms one finds that only 2_{II}^{-4}, $2_t^{\epsilon 4}$ and $2_t^{\epsilon 5}$ contain elements μ_1, μ_2 with the above stated norms.
Now we consider $n = 2$. Then $4_s^{\pm 2}$ is generated by two anisotropic elements γ, β of order 4 orthogonal to each other and $\mathrm{q}(2\gamma + 2\beta) = 0 \bmod 1$ and $\mathrm{q}(2\gamma) = \mathrm{q}(2\beta) = 1/2 \bmod 1$. And so the other component must not contain any non-trivial element μ of norm 0 or 1/2. This is the case only for 0 and $2_t^{\pm 1}$.
Finally, if $n \ge 3$, $4_s^{\pm 3}$ is generated by three anisotropic elements γ, β, μ of order 4 all pairwise orthogonal and $\mathrm{q}(\gamma) = s_1/8 \bmod 1$, $\mathrm{q}(\beta) = s_2/8 \bmod 1$ and $\mathrm{q}(\mu) = s_3/8 \bmod 1$. But then $2\gamma + 2\beta$ and $2\gamma + 2\mu$ generate an isotropic subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. This concludes the proof. □

Similarly, we prove the result for $(\mathbb{Z}/2\mathbb{Z})^3$ and get

Lemma 2.3.15. *Let $\mathcal{D}_3'$ be the set of discriminant forms D such that D is a sum of components of type $2_{II}^{\pm n}$, $2_t^{\pm n}$ and $4_s^{\pm m}$ and D contains no isotropic subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. Then $\mathcal{D}_3'$ consists of the following discriminant forms:*

$$\begin{aligned} &0,\ 2_{II}^{\pm 2},\ 2_{II}^{\pm 4},\ 2_{II}^{-6}, \\ &2_t^{\pm n},\ \textit{where } n \le 5 \\ &2_t^{\epsilon 6},\ \textit{where } \epsilon e(t/8) \neq 1 \\ &2_t^{\epsilon 7},\ \textit{where } \epsilon \left(\frac{t}{2}\right) = -1 \\ &4_s^{\pm 1},\ 2_{II}^{\pm 2} 4_s^{\pm 1},\ 2_{II}^{\pm 4} 4_s^{\pm 1}, \\ &2_t^{\pm n} 4_s^{\pm 1},\ \textit{where } n \le 5 \\ &4_s^{\pm 2},\ 2_{II}^{\pm 2} 4_s^{\pm 2} \\ &2_t^{\pm n} 4_s^{\pm 2},\ \textit{where } n \le 3 \\ &4_s^{\pm 3},\ 2_t^{\pm 1} 4_s^{\pm 3}. \end{aligned}$$

Proof. Let D be a sum of components of type $2_{II}^{\pm n}$, $2_t^{\pm n}$ and $4_s^{\pm m}$. First we assume that D is of type $2_{II}^{\pm n}$. If $n = 2$ then $D \in \mathcal{D}_2' \subset \mathcal{D}_3'$, so assume that $n \ge 4$. Then

we can write $2_{II}^{\pm n} \cong 2_{II}^{+2} \oplus 2_{II}^{\pm(n-2)}$ and we must have $2_{II}^{\pm(n-2)} \in \mathcal{D}_2'$. So we get $2_{II}^{\pm 4}, 2_{II}^{-6} \in \mathcal{D}_3'$.
Now we assume that D has level 4. Then D is of type $2_t^{\epsilon n}$. If $D \in \mathcal{D}_1'$ then also $D \in \mathcal{D}_3'$. Otherwise D contains an isotropic element γ of order 2 and $\langle\gamma\rangle^\perp/\langle\gamma\rangle \cong 2_t^{\epsilon(n-2)}$ or $\langle\gamma\rangle^\perp/\langle\gamma\rangle \cong 2_{II}^{\epsilon'(n-2)}$, with $\epsilon' = \epsilon(-1)^{(t-4)/2}$. Now $D \in \mathcal{D}_3'$ if and only if $\langle\gamma\rangle^\perp/\langle\gamma\rangle \in \mathcal{D}_2'$. This shows that $D \in \mathcal{D}_3'$ when $n \le 5$. For $n = 6$ we see that $D \in \mathcal{D}_3$ if $\epsilon e(t/8) = e(\mathrm{sign}(D)/8) \neq 1$ and for $n = 7$ if $\epsilon\left(\frac{t}{2}\right) = -1$.
Now we assume that D has level 8 and $D = 2_{II}^{\pm m} 4_s^{\pm n}$ or $D = 2_t^{\pm m} 4_s^{\pm n}$. First we assume that $n = 1$. Then $4_s^{\pm 1}$ is generated by an element γ of order 4 and $\mathrm{q}(2\gamma) = 1/2 \mod 1$. So the other component must contain neither an isotropic subgroup H isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, nor an isotropic subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ with an additional element $\mu \in H^\perp$ of norm $1/2 \mod 1$. We have already seen that only

$$
\begin{aligned}
&0,\ 2_{II}^{\pm 2},\ 2_{II}^{\pm 4},\ 2_{II}^{-6},\\
&2_t^{\pm n},\ \text{where } n \le 5\\
&2_t^{\epsilon 6},\ \text{where } \epsilon e(t/8) \neq 1\\
&2_t^{\epsilon 7},\ \text{where } \epsilon\left(\frac{t}{2}\right) = -1
\end{aligned}
$$

contain no isotropic subgroup H isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. Going through this finite list of discriminant forms one finds that only 2_{II}^{-6}, $2_t^{\epsilon 6}$ and $2_t^{\epsilon 7}$ contain a subgroup H and μ of norm $1/2 \mod 1$ as described above.
Next assume that $n = 2$. Then $4_s^{\pm 2}$ is generated by two elements γ, β of order 4 and $\mathrm{q}(2\gamma + 2\beta) = 0 \mod 1$ and $\mathrm{q}(2\gamma) = \mathrm{q}(2\beta) = 1/2 \mod 1$. So the other component must not contain any isotropic subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$, but also no μ_1, μ_2 with $(\mu_1, \mu_2) = 0 \mod 1$ and $\mathrm{q}(\mu_1) = \mathrm{q}(\mu_2) = 1/2 \mod 1$. But we have already seen in the proof of the previous lemma that this holds exactly for 0, $2_{II}^{\pm 2}$ and $2_t^{\pm n}$, where $n \le 3$.
If $n = 3$, then $4_s^{\pm 3}$ is generated by three elements γ, β, μ of order 4 and $\mathrm{q}(2\gamma + 2\beta) = \mathrm{q}(2\gamma + 2\mu) = 0 \mod 1$ and $\mathrm{q}(2\gamma) = 1/2 \mod 1$. So the other component must not contain any non-trivial element of norm 0 or $1/2 \mod 1$. Therefore, we get $4_s^{\pm 3}$ and $2_t^{\pm 1} 4_s^{\pm 3}$.
If $n \ge 4$, $4_s^{\pm 4}$ is generated by four anisotropic elements $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ of order 4 all pairwise orthogonal with $\mathrm{q}(\gamma_i) = s_i/8 \mod 1$. Then $2\gamma_1 + 2\gamma_2$, $2\gamma_2 + 2\gamma_3$ and $2\gamma_3 + 2\gamma_4$ generate an isotropic subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. This concludes the proof. □

Now we can say under which conditions a 2-adic discriminant form contains no isotropic subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$.

Proposition 2.3.16. *Let D be a discriminant form. We choose a Jordan decomposition of D and write $D = A \oplus B$, where A denotes the sum over the components of type $2_{II}^{\pm n}$, $2_t^{\pm n}$ and $4_s^{\pm m}$ and B the sum over the remaining components. Then D contains no isotropic subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$ if and only if B has rank r with $0 \leq r < 3$ and $A \in \mathcal{D}'_{3-r}$.*

Proof. The elements of order 2 in B generate an isotropic subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$, so D contains no isotropic subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$ if and only if A contains no isotropic subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{3-r}$. The previous lemmas prove the proposition. ☐

Before we can show for which 2-adic discriminant forms one has $\mathrm{im}(\uparrow) = \mathbb{C}[D]$, we need

Lemma 2.3.17. *Let A be a discriminant form isomorphic to a sum of components of type $2_{II}^{\pm n}$, $2_t^{\pm n}$ and $4_s^{\pm m}$. Let $q, \tilde{q} \geq 8$ be powers of 2, $t, \tilde{t} \in \{1,3,5,7\}$ and $\epsilon = \left(\frac{t}{2}\right), \tilde{\epsilon} = \left(\frac{\tilde{t}}{2}\right)$. Then $\mathrm{im}(\uparrow) = \mathbb{C}[A \oplus q_t^{\epsilon}]$ if and only if $\mathrm{im}(\uparrow) = \mathbb{C}[A \oplus \tilde{q}_{\tilde{t}}^{\tilde{\epsilon}}]$.*

Proof. Suppose that $\mathrm{im}(\uparrow) = \mathbb{C}[A \oplus q_t^{\epsilon}]$ and let $\tilde{\gamma} \in A \oplus \tilde{q}_{\tilde{t}}^{\tilde{\epsilon}}$ be arbitrary. We assume that q_t^{ϵ} is generated by an element β with $\mathrm{q}(\beta) = \frac{t}{2q} \bmod 1$ and $\tilde{q}_{\tilde{t}}^{\tilde{\epsilon}}$ is generated by an element $\tilde{\beta}$ with $\mathrm{q}(\tilde{\beta}) = \frac{\tilde{t}}{2\tilde{q}} \bmod 1$. Write $\tilde{\gamma} = \alpha + x\tilde{\beta}$ with $\alpha \in A$ and $x \in \mathbb{Z}$. By assumption, for $\gamma = \alpha + x\beta \in A \oplus q_t^{\epsilon}$ we have $e^{\gamma} \in \mathrm{im}(\uparrow)$. Note that the elements in the connected component of γ in $G_{A \oplus q_t^{\epsilon}}$ are of the form $\gamma + \sum_i \mu_i$, where $\mu_i \in A \oplus q_t^{\epsilon}$ is isotropic of order 2. Similarly the elements in the connected component of $\tilde{\gamma}$ in $G_{A \oplus \tilde{q}_{\tilde{t}}^{\tilde{\epsilon}}}$ are of the form $\tilde{\gamma} + \sum_i \tilde{\mu}_i$, where $\tilde{\mu}_i \in A \oplus \tilde{q}_{\tilde{t}}^{\tilde{\epsilon}}$ is isotropic of order 2. The map

$$\mu = \mu' + (q/2)\beta \mapsto \tilde{\mu} = \mu' + (\tilde{q}/2)\tilde{\beta}$$

defines a bijection from the isotropic elements of order 2 in $A \oplus q_t^{\epsilon}$ to those in $A \oplus \tilde{q}_{\tilde{t}}^{\tilde{\epsilon}}$. For all isotropic $\mu_1, \mu_2 \in A \oplus q_t^{\epsilon}$ of order 2 we have

$$\begin{aligned}
(\tilde{\gamma}, \tilde{\mu}_1) &= (\alpha, \mu_1') + x(\tilde{q}/2)\frac{\tilde{t}}{\tilde{q}} \mod 1 \\
&= (\alpha, \mu_1') + x\frac{\tilde{t}}{2} \mod 1 \\
&= (\alpha, \mu_1') + x(q/2)\frac{t}{q} \mod 1 \\
&= (\gamma, \mu_1) \mod 1
\end{aligned}$$

and

$$\begin{aligned}(\tilde{\mu_1}, \tilde{\mu_2}) &= (\mu_1', \mu_2') + (\tilde{q}/2)(\tilde{q}/2)\frac{\tilde{t}}{\tilde{q}} \mod 1 \\ &= (\mu_1', \mu_2') \mod 1 \\ &= (\mu_1, \mu_2) \mod 1.\end{aligned}$$

This shows that the connected component of γ and the connected component of $\tilde{\gamma}$ are isomorphic graphs and hence $e^{\tilde{\gamma}} \in \mathrm{im}(\uparrow)$ as well. Reversing the roles of $A \oplus q_t^\epsilon$ and $A \oplus \tilde{q}_{\tilde{t}}^{\tilde{\epsilon}}$ proves the lemma. □

As for the p-adic case we refine the result of Proposition 2.3.16 to get a precise statement when $\mathrm{im}(\uparrow) = \mathbb{C}[D]$.

Theorem 2.3.18. *Let $\mathcal{D}_1 = \mathcal{D}_1'$, i.e. the set of discriminant forms consisting of*

$$\begin{aligned}&0,\ 2_{II}^{-2},\ 2_t^{\pm 1}, \\ &2_t^{\pm 2},\ \textit{where } t = 2 \bmod 4, \\ &2_t^{\epsilon 3},\ \textit{where } \epsilon\left(\frac{t}{2}\right) = -1, \\ &4_t^{\pm 1},\ 2_t^{\pm 1}4_s^{\pm 1},\end{aligned}$$

let $\mathcal{D}_2$ be the set of discriminant forms consisting of

$$\begin{aligned}&0,\ 2_{II}^{\pm 2}, \\ &2_t^{\pm n},\ \textit{where } n \leq 3 \\ &2_t^{\epsilon 4},\ \textit{where } \epsilon e(t/8) \neq 1 \\ &4_s^{\pm 1},\ 2_{II}^{\pm 2}4_s^{\pm 1}, \\ &2_t^{\pm n}4_s^{\pm 1},\ \textit{where } n \leq 3 \\ &4_s^{\pm 2},\ 2_t^{\pm 1}4_s^{\pm 2}\end{aligned}$$

and let $\mathcal{D}_3$ be the set of discriminant forms consisting of

$$\begin{aligned}&0,\ 2_{II}^{\pm 2},\ 2_{II}^{\pm 4}, \\ &2_t^{\pm n},\ \textit{where } n \leq 5 \\ &2_t^{\pm 6},\ \textit{where } t = 2 \bmod 4 \\ &4_s^{\pm 1},\ 2_{II}^{\pm 2}4_s^{\pm 1},\ 2_{II}^{\pm 4}4_s^{\pm 1}, \\ &2_t^{\pm n}4_s^{\pm 1},\ \textit{where } n \leq 5 \\ &4_s^{\pm 2},\ 2_{II}^{\pm 2}4_s^{\pm 2} \\ &2_t^{\pm n}4_s^{\pm 2},\ \textit{where } n \leq 3 \\ &4_s^{\pm 3},\ 2_t^{\pm 1}4_s^{\pm 3}.\end{aligned}$$

Let D be a discriminant form of level a power of 2. We choose a Jordan decomposition of D and write $D = A \oplus B$, where A denotes the sum over the components of type $2_{II}^{\pm n}$, $2_t^{\pm n}$ and $4_s^{\pm m}$ and B the sum over the remaining components. Then $\mathrm{im}(\uparrow) = \mathbb{C}[D]$ *unless B has rank r with $0 \leq r < 3$ and $A \in \mathcal{D}_{3-r}$.*

Proof. If D contains an isotropic subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, then by Corollary 2.3.3 $\mathrm{im}(\uparrow) = \mathbb{C}[D]$. Therefore, we consider the discriminant forms described in Proposition 2.3.16. We show that for those discriminant forms described in Theorem 2.3.18, there is an element $e^\gamma \notin \mathrm{im}(\uparrow)$. For those discriminant forms not in Theorem 2.3.18, we show that $\mathrm{im}(\uparrow) = \mathbb{C}[D]$.
First we assume that $r = 2$ and that $A \in \mathcal{D}_1'$. If $B = q_{II}^{\pm 2}$ for q some power of 2 and $\gamma \in B$ of order q, then $\gamma^\perp = \langle \beta \rangle \oplus A$, where $\beta \in B$ is some element of order q. Therefore, the only isotropic element of order two orthogonal to γ is $(q/2) \cdot \beta$. If on the other hand B is the sum of odd 2-adic components and generated by two anisotropic elements γ and β orthogonal to each other, then $\gamma^\perp = \langle \beta \rangle \oplus A$ and so the only isotropic element of order two orthogonal to γ is $(\mathrm{ord}(\beta)/2) \cdot \beta$. In both cases Lemma 2.3.12 implies $e^\gamma \notin \mathrm{im}(\uparrow)$. Hence, $\mathcal{D}_1 = \mathcal{D}_1'$.
Now we assume that $r \leq 1$. If $r = 1$, because of Lemma 2.3.17, we can assume that $B = 8_1^{+1}$. Therefore, we only need to check for a finite number of discriminant forms D whether $\mathrm{im}(\uparrow) = \mathbb{C}[D]$, i.e. whether G_D contains no bipartite component. This can be done quickly by a computer. This was done using the computational algebra system Magma [10]. □

2.4 Main theorem on isotropic lifts

Now we can finally prove the main theorem. We first define what it means for a discriminant form to be of small type.

First let D be a discriminant form of level a power of p for some prime p. If p is odd then we say that D is of *small type* if one of the following conditions holds:

(i) D has rank two or less.

(ii) D has rank three and at least one Jordan component is of level p.

(iii) D has rank four and is of type $p^{-\epsilon 2} q_1^{\pm 1} q_2^{\pm 1}$, where $\epsilon = \left(\frac{-1}{p}\right)$ and q_1, q_2 are powers of p and can also be p.

(iv) D has rank five and is of level p.

If $p = 2$ we choose a Jordan decomposition of D and write $D = A \oplus B$, where A denotes the sum over the components of type $2_{II}^{\pm n}$, $2_t^{\pm n}$ and $4_s^{\pm m}$ and B the sum over the remaining components. Then we say that D is of small type if B has rank r with $0 \leq r < 3$ and $A \in \mathcal{D}_{3-r}$, where $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ are as defined in Theorem 2.3.18. Recall that the components $2_{II}^{\pm n}$, $2_t^{\pm n}$ and $4_s^{\pm m}$ are exactly those containing anisotropic elements of order 2.
Now let D be a discriminant form of arbitrary level $N = \prod_{p|N} p^{\nu_p}$. We say that D is of small type if for all $p \mid N$ the p-subgroups $D_{p^{\nu_p}}$ of D are of small type.

We remark that any discriminant form of rank ≥ 7 is not of small type and any discriminant form of odd level and rank ≥ 6 is not of small type. We get

Theorem 2.4.1. *Let D be a discriminant form. Then all modular forms for the Weil representation of D are linear combinations of modular forms of the form $\uparrow_H(f)$, where $H \subset D$ is an isotropic subgroup such that $H^\perp/H$ is of small type and f is a modular form for the Weil representation of $H^\perp/H$. For any discriminant form of small type there exist modular forms which are not linear combinations of isotropically lifted modular forms.*

Proof. If D is of small type then $\mathrm{im}(\uparrow) \subsetneq \mathbb{C}[D]$ by the previous two subsections. From Theorem 2.2.4 we know that in this case there exist modular forms which are not linear combinations of isotropically lifted modular forms. To show that for a discriminant form not of small type we can write all modular form as linear combinations of modular forms lifted from discriminant forms of small type we use induction on $|D|$. If D is trivial then D is of small type and there is nothing to prove. Now assume that $|D| > 1$ and that the assertion holds for discriminant forms of order smaller than $|D|$. If D is of small type then, again, there is nothing to prove. Otherwise we have seen in the previous two subsections that $\mathrm{im}(\uparrow) = \mathbb{C}[D]$. Again using Theorem 2.2.4 we know that all modular forms for D for all weights $k \in \frac{1}{2}\mathbb{Z}$ are linear combinations of isotropically lifted modular forms. Using the induction hypothesis and the fact that the isotropic lifts are transitive we get the result. $\square$

We remark that in his Ph.D. Thesis [71] Werner showed that for a discriminant form D of level N all modular forms are linear combinations of isotropically lifted modular forms if $|D| \geq N^9$. In contrast to Werner's result Theorem 2.4.1 is sharp.

Chapter 3

Invariants of the Weil representation

In this chapter we will investigate the space of invariants for the Weil representation. In particular, we will show that all invariants are induced from five fundamental invariants, specializing the main theorem of the previous chapter.

This chapter is based on joint work with Nils Scheithauer (cf. [53]).

3.1 General results on invariants

Let D be a discriminant form. Recall that the non-trivial element in the kernel of the covering map $\mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$ acts as $(-1)^{\mathrm{sign}(D)}$, so that the space of invariants $\mathbb{C}[D]^{\mathrm{Mp}_2(\mathbb{Z})}$ is trivial if D has odd signature and when D has even signature the Weil representation ρ_D descends to a representation of $\mathrm{SL}_2(\mathbb{Z})$. So let from now on D be a discriminant form of even signature. Throughout this chapter we write $\Gamma = \Gamma^{(1)} = \mathrm{SL}_2(\mathbb{Z})$. We define the space of invariants by

$$\mathbb{C}[D]^\Gamma \coloneqq \{v \in \mathbb{C}[D] \mid \rho_D(M)v = v \text{ for all } M \in \mathrm{SL}_2(\mathbb{Z})\}.$$

We first describe some general properties of $\mathbb{C}[D]^\Gamma$.

We denote in this chapter the set of isotropic elements in D by I. Let $v = \sum_{\gamma\in D} v_\gamma e^\gamma$ be an invariant of Γ. Then the T-invariance implies that $v_\gamma = 0$ if $\gamma \notin I$. Hence, $\dim \mathbb{C}[D]^\Gamma \leq |I|$. We give an exact formula below.

We recall some properties of $\Gamma(N)$. The group $\Gamma(N)$ is a normal subgroup of Γ and has index

$$|\Gamma(N)\backslash\Gamma| = N^3 \prod_{p|N} (1 - 1/p^2)$$

in Γ. The number of classes of cusps is

$$|\Gamma(N)\backslash P| = \begin{cases} 3 & \text{if } N = 2\,, \\ (N^2/2)\prod_{p|N}(1-1/p^2) & \text{if } N \geq 3 \end{cases}$$

where $P = \mathbb{Q}\cup\{\infty\}$. The (classes of) cusps are parametrised by the elements (a,c) of order N in $(\mathbb{Z}/N\mathbb{Z})^2$ if $N=2$ and by the pairs $\pm(a,c)$ of elements of order N in $(\mathbb{Z}/N\mathbb{Z})^2$ if $N\geq 3$ (see Lemma 3.8.4 in [21]). Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Then the cosets of $\Gamma(N)\backslash\Gamma$ sending ∞ to a/c can be represented by MT^n if $N=2$ and by $\pm MT^n$ if $N\geq 3$ where in both cases n ranges over a complete set of residues modulo N.

Now we describe the projection on the subspace of invariants. We define the map

$$\mathrm{inv}_D : \mathbb{C}[D] \to \mathbb{C}[D]$$

by

$$\mathrm{inv}_D(e^\gamma) = \frac{1}{|\Gamma(N)\backslash\Gamma|} \sum_{M\in\Gamma(N)\backslash\Gamma} \rho_D(M^{-1})e^\gamma .$$

It maps onto the subspace of invariants $\mathbb{C}[D]^\Gamma$. Since $\Gamma(N)$ is normal in Γ and ρ_D is unitary, we have

$$\langle \mathrm{inv}_D(v), w\rangle = \langle v, \mathrm{inv}_D(w)\rangle$$

for all $v,w\in\mathbb{C}[D]$. Let $v=\sum_{\gamma\in D} v_\gamma e^\gamma \in \mathbb{C}[D]^\Gamma$. Then $\langle v, \mathrm{inv}_D(e^\gamma)\rangle = v_\gamma$. This implies $\mathrm{inv}_D(e^\gamma)=0$ if $\gamma\notin I$. Furthermore, inv_D commutes with $\rho_D(M)$ for all $M\in\Gamma$.

We can calculate $\mathrm{inv}_D(e^\gamma)$ as follows.

Theorem 3.1.1. *Let D be a discriminant form of even signature and level dividing N and $\gamma\in I$. Then*

$$\mathrm{inv}_D(e^\gamma) = \sum_{s\in\Gamma(N)\backslash P} \mathrm{inv}_D(e^\gamma)_s$$

with

$$\mathrm{inv}_D(e^\gamma)_s \;=\; \xi(M^{-1})\,\frac{N}{|\Gamma(N)\backslash\Gamma|}\,\frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\mu\in(a\gamma+D^{c*})\cap I} e(-d\,\mathrm{q}_c(\mu-a\gamma))e(-b(\mu,\gamma))e^\mu$$

if $N=2$ and

$$\mathrm{inv}_D(e^\gamma)_s \;=\; \xi(M^{-1})\,\frac{N}{|\Gamma(N)\backslash\Gamma|}\,\frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\mu\in(a\gamma+D^{c*})\cap I} e(-d\,\mathrm{q}_c(\mu-a\gamma))e(-b(\mu,\gamma))\big\{e^\mu + e(\mathrm{sign}(D)/4)e^{-\mu}\big\}$$

if $N\geq 3$ where in both cases $M=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is any matrix in Γ such that $M\infty = s$.

Proof. We can write

$$\mathrm{inv}_D(e^\gamma) = \sum_{s\in\Gamma(N)\backslash P} \mathrm{inv}_D(e^\gamma)_s$$

with

$$\mathrm{inv}_D(e^\gamma)_s = \frac{1}{|\Gamma(N)\backslash\Gamma|} \sum_{\substack{M\in\Gamma(N)\backslash\Gamma \\ M\infty=s}} \rho_D(M^{-1})e^\gamma$$

Suppose $N \geq 3$. Let $s \in P$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ such that $M\infty = s$. Then

$$\begin{aligned}
\mathrm{inv}_D(e^\gamma)_s &= \frac{1}{|\Gamma(N)\backslash\Gamma|} \sum_{n\in\mathbb{Z}/N\mathbb{Z}} \left\{\rho_D((MT^n)^{-1})e^\gamma + \rho_D((-MT^n)^{-1})e^\gamma\right\} \\
&= \frac{1}{|\Gamma(N)\backslash\Gamma|} \sum_{n\in\mathbb{Z}/N\mathbb{Z}} \rho_D(T^{-n})\rho_D(M^{-1})\{e^\gamma + e(\mathrm{sign}(D)/4)e^{-\gamma}\} \\
&= \xi(M^{-1})\,\frac{1}{|\Gamma(N)\backslash\Gamma|}\,\frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\mu\in a\gamma+D^{c*}} e(-d\,\mathrm{q}_c(\mu-a\gamma))e(-b(\mu,\gamma)) \\
&\qquad \sum_{n\in\mathbb{Z}/N\mathbb{Z}} \rho_D(T^{-n})\{e^\mu + e(\mathrm{sign}(D)/4)e^{-\mu}\} \\
&= \xi(M^{-1})\,\frac{N}{|\Gamma(N)\backslash\Gamma|}\,\frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\mu\in(a\gamma+D^{c*})\cap I} e(-d\,\mathrm{q}_c(\mu-a\gamma))e(-b(\mu,\gamma)) \\
&\qquad \{e^\mu + e(\mathrm{sign}(D)/4)e^{-\mu}\},
\end{aligned}$$

where we used the explicit formula (1.2.1) for the Weil representation, $\gamma \in I$ and

$$\sum_{n\in\mathbb{Z}/N\mathbb{Z}} \rho_D(T^{-n})e^\mu = 0$$

if $\mu \notin I$. For $N = 2$ we just drop the second sum. □

The dimension of the subspace of invariants is given by the trace of the linear map inv_D, i.e.

$$\dim \mathbb{C}[D]^\Gamma = \mathrm{tr}\ \mathrm{inv}_D = \sum_{\gamma\in I} \langle \mathrm{inv}_D(e^\gamma), e^\gamma\rangle = \sum_{\gamma\in I}\ \sum_{s\in\Gamma(N)\backslash P} \langle \mathrm{inv}_D(e^\gamma)_s, e^\gamma\rangle\,.$$

The previous theorem implies

Theorem 3.1.2. *Let D be a discriminant form of even signature and level dividing N. Then*

$$\dim \mathbb{C}[D]^\Gamma = \sum_{s\in\Gamma(N)\backslash P} d_s$$

with

$$d_s = \xi(M^{-1}) \frac{N}{|\Gamma(N)\backslash\Gamma|} \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\substack{\gamma\in I \\ (1-a)\gamma\in D^{c*}}} e(-d\,\mathrm{q}_c((1-a)\gamma))$$

if $N=2$ *and*

$$d_s = \xi(M^{-1}) \frac{N}{|\Gamma(N)\backslash\Gamma|} \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \left\{ \sum_{\substack{\gamma\in I \\ (1-a)\gamma\in D^{c*}}} e(-d\,\mathrm{q}_c((1-a)\gamma)) + e(\mathrm{sign}(D)/4) \sum_{\substack{\gamma\in I \\ (1+a)\gamma\in D^{c*}}} e(-d\,\mathrm{q}_c((1+a)\gamma)) \right\}$$

if $N \geq 3$*, where in both cases* $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *is any matrix in* Γ *such that* $M\infty = s$*.*

We describe some properties of the invariants of ρ_D and the projection inv_D.

Proposition 3.1.3. *Let* D *be a discriminant form of even signature and level dividing* N*. Let* $v = \sum_{\gamma\in D} v_\gamma e^\gamma \in \mathbb{C}[D]^\Gamma$*. Then*

$$v_\gamma = \chi_D(a) v_{a\gamma}$$

for all $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ *and* $\gamma \in D$*. If* χ_D *is non-trivial and* H *is a subgroup of* D*, then*

$$\sum_{\gamma\in H} v_\gamma = 0\,.$$

Proof. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Then

$$v = \rho_D(M)v = \sum_{\gamma\in I} v_\gamma \rho_D(M) e^\gamma = \chi_D(a) \sum_{\gamma\in I} v_\gamma e^{d\gamma} = \chi_D(a) \sum_{\gamma\in I} v_{a\gamma} e^\gamma\,.$$

For the second statement note that H decomposes into orbits under the action of $(\mathbb{Z}/N\mathbb{Z})^\times$ and

$$\sum_{(a,N)=1} v_{a\gamma} = \sum_{(a,N)=1} \chi_D(a) v_\gamma = v_\gamma \sum_{(a,N)=1} \chi_D(a) = 0\,.$$

This proves the proposition. □

Proposition 3.1.4. *Let* D *be a discriminant form of even signature with non-trivial* χ_D*. Then*

$$\mathrm{inv}_D(e^0) = 0\,.$$

Proof. Since 0 is a subgroup of D, we have $\langle v, \mathrm{inv}_D(e^0)\rangle = v_0 = 0$ for all $v = \sum_{\gamma\in D} v_\gamma e^\gamma \in \mathbb{C}[D]^\Gamma$. Hence, $\mathrm{inv}_D(e^0) = 0$. □

Proposition 3.1.5. *Let D be a discriminant form of even signature and $\gamma \in I^\perp$. Then* $\mathrm{inv}_D(e^\gamma) = \mathrm{inv}_D(e^0)$.

Proof. Let $v = \sum_{\beta\in D} v_\beta e^\beta \in \mathbb{C}[D]^\Gamma$. Then the invariance of v under J implies

$$\begin{aligned} v_\gamma &= \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta\in D} v_\beta\, e((\gamma,\beta)) = \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta\in I} v_\beta\, e((\gamma,\beta)) \\ &= \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta\in I} v_\beta = \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta\in D} v_\beta = v_0\,. \end{aligned}$$

It follows $\langle v, \mathrm{inv}_D(e^\gamma)\rangle = v_\gamma = v_0 = \langle v, \mathrm{inv}_D(e^0)\rangle$. □

Let D be a discriminant form of even signature which contains no non-trivial isotropic element, i.e. $I = \{0\}$. Then $I^\perp = D$. If in addition D is non-trivial, then the proposition implies that $\mathrm{inv}_D(e^0) = 0$ and $\dim(\mathbb{C}[D]^\Gamma) = 0$. (Choose $\gamma \in D\backslash\{0\}$. Then γ is non-isotropic so that $\mathrm{inv}_D(e^0) = \mathrm{inv}_D(e^\gamma) = 0$.)

Proposition 3.1.6. *Let D be a discriminant form of even signature with non-trivial χ_D and $\gamma \in D$ such that $2\gamma \in I^\perp$. Then* $\mathrm{inv}_D(e^\gamma) = 0$.

Proof. Let $v = \sum_{\beta\in D} v_\beta e^\beta \in \mathbb{C}[D]^\Gamma$. The invariance of v under J and Proposition 3.1.3 give

$$\begin{aligned} v_\gamma &= \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta\in D} v_\beta\, e((\gamma,\beta)) = \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta\in I} v_\beta\, e((\gamma,\beta)) \\ &= \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \Bigg(\sum_{\substack{\beta\in I \\ (\beta,\gamma)=0 \bmod 1}} v_\beta - \sum_{\substack{\beta\in I \\ (\beta,\gamma)=1/2 \bmod 1}} v_\beta \Bigg) \\ &= \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \Bigg(2 \sum_{\substack{\beta\in I \\ (\beta,\gamma)=0 \bmod 1}} v_\beta - \sum_{\beta\in I} v_\beta \Bigg) \\ &= -v_0 + 2\,\frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta\in\gamma^\perp} v_\beta = 0 \end{aligned}$$

because 0 and $\gamma^\perp$ are subgroups of D. □

Proposition 3.1.7. *Let D be a discriminant form of even signature and level dividing N. Suppose $(N,5) = 1$ and $\chi_D(5) = -1$. Let $\gamma \in D$ such that $4\gamma \in I^\perp$. Then* $\mathrm{inv}_D(e^\gamma) = 0$.

Proof. For $v = \sum_{\beta\in D} v_\beta e^\beta \in \mathbb{C}[D]^\Gamma$ we have

$$v_\gamma = \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{j=0}^{3} e(j/4) \sum_{\substack{\beta\in I \\ (\beta,\gamma)=j/4 \bmod 1}} v_\beta\,.$$

The sets $\{\beta \in I \mid (\beta, \gamma) = j/4 \bmod 1\}$ are invariant under multiplication by 5. On the other hand $v_{5\beta} = \chi_D(5) v_\beta = -v_\beta$ for all $\beta \in D$ by Proposition 3.1.3. It follows

$$2 \sum_{\substack{\beta \in I \\ (\beta,\gamma) = j/4 \bmod 1}} v_\beta = \sum_{\substack{\beta \in I \\ (\beta,\gamma) = j/4 \bmod 1}} (v_\beta + v_{5\beta}) = 0\,.$$

This implies the statement. □

Note that the condition of the proposition is satisfied for example for 2-adic discriminant forms D of even signature such that $|D|$ is not a square.

3.2 Discriminant forms of prime level

In this section we calculate the projection on the subspace of invariants and the dimension of this space explicitly for discriminant forms of prime level.

We start with the case that p is odd.

Theorem 3.2.1. *Let p be an odd prime and D a discriminant form of type $p^{\epsilon n}$. Let $\gamma \in I$. Then*

$$\mathrm{inv}_D(e^\gamma) = \epsilon \left(\frac{-1}{p}\right)^{n/2} \frac{1}{p^2-1} \frac{1}{p^{(n-2)/2}} \left\{ \sum_{\mu \in (\gamma^\perp \cap I)} p\, e^\mu - \sum_{\mu \in I} e^\mu \right\} + \frac{1}{p^2-1} \sum_{a \in (\mathbb{Z}/p\mathbb{Z})^\times} e^{a\gamma}$$

if n is even and

$$\mathrm{inv}_D(e^\gamma) = \epsilon \left(\frac{-1}{p}\right)^{(n+1)/2} \left(\frac{2}{p}\right) \frac{1}{p^2-1} \frac{1}{p^{(n-3)/2}} \sum_{\mu \in I} \left(\frac{p(\mu,\gamma)}{p}\right) e^\mu + \frac{1}{p^2-1} \sum_{a \in (\mathbb{Z}/p\mathbb{Z})^\times} \left(\frac{a}{p}\right) e^{a\gamma}$$

if n is odd.

Proof. The cusps of $\Gamma(p)$ are represented by the pairs $\pm(a,c) \in (\mathbb{Z}/p\mathbb{Z})^2 \backslash \{(0,0)\}$. If $(c,p) = 1$, we can choose any $d \in \mathbb{Z}/p\mathbb{Z}$ and define $b = c^{-1}(ad-1)$ to obtain a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ (c^{-1} denotes the inverse of c modulo p). Let M_s be any lift of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to Γ (recall that the projection $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ is surjective).

Then

$$\operatorname{inv}_D(e^\gamma)_s = \xi(M_s^{-1})\,\frac{1}{p^2-1}\,\frac{1}{p^{n/2}} \sum_{\mu\in(a\gamma+D^{c*})\cap I} e(-d\,\mathrm{q}_c(\mu-a\gamma))e(-b(\mu,\gamma))\big\{e^\mu+e(\operatorname{sign}(D)/4)e^{-\mu}\big\}.$$

Taking $d=0 \bmod p$ and using the explicit formula for $\xi(M_s^{-1})$ given in [60] we obtain

$$\operatorname{inv}_D(e^\gamma)_s = e(\operatorname{sign}(D)/8)\left(\frac{c}{|D|}\right)\frac{1}{p^2-1}\,\frac{1}{p^{n/2}} \sum_{\mu\in I} e(c^{-1}(\mu,\gamma))\{e^\mu+e(\operatorname{sign}(D)/4)e^{-\mu}\}.$$

Recall that in [60] the dual Weil representation was used.
If $c=0 \bmod p$, we choose a matrix $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ and lift it to a matrix M_s in Γ. Then

$$\operatorname{inv}_D(e^\gamma)_s = \left(\frac{a}{|D|}\right)\frac{1}{p^2-1}\,\{e^{a\gamma}+e(\operatorname{sign}(D)/4)e^{-a\gamma}\}.$$

Summing over all cusps of $\Gamma(p)$ we get

$$\begin{aligned}\operatorname{inv}_D(e^\gamma) = \frac{1}{2}\,e(\operatorname{sign}(D)/8)\,\frac{1}{p^2-1}\,\frac{1}{p^{(n-2)/2}}\sum_{\mu\in I}&\{e^\mu+e(\operatorname{sign}(D)/4)e^{-\mu}\}\\ &\sum_{c\in(\mathbb{Z}/p\mathbb{Z})^\times}\left(\frac{c}{|D|}\right)e(c(\mu,\gamma))\\ &+\frac{1}{2}\,\frac{1}{p^2-1}\sum_{a\in(\mathbb{Z}/p\mathbb{Z})^\times}\left(\frac{a}{|D|}\right)\{e^{a\gamma}+e(\operatorname{sign}(D)/4)e^{-a\gamma}\}.\end{aligned}$$

If n is even, then $e(\operatorname{sign}(D)/8)=\epsilon\big(\frac{-1}{p}\big)^{n/2}$ (see the proof of Theorem 7.1 in [59]) and

$$\sum_{c\in(\mathbb{Z}/p\mathbb{Z})^\times}\left(\frac{c}{|D|}\right)e(c(\mu,\gamma)) = \begin{cases} p-1 & \text{if } (\mu,\gamma)=0 \mod 1,\\ -1 & \text{otherwise}\end{cases}$$

so that

$$\operatorname{inv}_D(e^\gamma) = \epsilon\left(\frac{-1}{p}\right)^{n/2}\frac{1}{p^2-1}\,\frac{1}{p^{(n-2)/2}}\left\{\sum_{\mu\in(\gamma^\perp\cap I)} p\,e^\mu - \sum_{\mu\in I} e^\mu\right\} + \frac{1}{p^2-1}\sum_{a\in(\mathbb{Z}/p\mathbb{Z})^\times} e^{a\gamma}\,.$$

If n is odd, the statement follows from

$$e(\mathrm{sign}(D)/8) = \epsilon \left(\frac{2}{p}\right) \begin{cases} 1 & \text{if } p = 1 \bmod 4\,, \\ (-1)^{(n+1)/2}(-i) & \text{if } p = 3 \bmod 4 \end{cases}$$

and

$$\sum_{c \in (\mathbb{Z}/p\mathbb{Z})^\times} \left(\frac{c}{p}\right) e(c(\mu,\gamma)) = \left(\frac{p(\mu,\gamma)}{p}\right) \sqrt{p} \begin{cases} 1 & \text{if } p = 1 \bmod 4\,, \\ i & \text{if } p = 3 \bmod 4 \end{cases}$$

(see Theorem 1.2.4 in [2]). □

Note that for $n = 1$ or $n = 2$ and $\epsilon\left(\frac{-1}{p}\right) = -1$ we have $I = \{0\}$ and $\mathrm{inv}_D(e^\gamma) = 0$ for all $\gamma \in D$. The first formula in the theorem extends to discriminant forms of level 2.

Theorem 3.2.2. *Let D be a discriminant form of type $2_{II}^{\epsilon n}$ with n even and $\gamma \in I$. Then*

$$\mathrm{inv}_D(e^\gamma) = \epsilon\, \frac{1}{3}\, \frac{1}{2^{(n-2)/2}} \left\{ \sum_{\mu \in (\gamma^\perp \cap I)} 2\, e^\mu - \sum_{\mu \in I} e^\mu \right\} + \frac{1}{3}\, e^\gamma\,.$$

Next we calculate the dimensions of the subspace of invariants.

Theorem 3.2.3. *Let p be an odd prime and D a discriminant form of type $p^{\epsilon n}$. Then*

$$\dim \mathbb{C}[D]^\Gamma = \frac{p^{n-1} - p}{p^2 - 1} + \epsilon \left(\frac{-1}{p}\right)^{n/2} p^{(n-2)/2} + 1$$

if n is even and

$$\dim \mathbb{C}[D]^\Gamma = \frac{p^{n-1} - 1}{p^2 - 1}$$

if n is odd.

Proof. This can be proved directly by using Theorem 3.1.2 or by means of Theorem 3.2.1. We describe the second approach for n even. For $\gamma \in I$ we have

$$\langle \mathrm{inv}_D(e^\gamma), e^\gamma \rangle = \epsilon \left(\frac{-1}{p}\right)^{n/2} \frac{1}{p^2-1}\, \frac{1}{p^{(n-2)/2}}\, (p-1) + \frac{1}{p^2-1} \begin{cases} 1 & \text{if } \gamma \neq 0, \\ p-1 & \text{if } \gamma = 0 \end{cases}$$

so that

$$\begin{aligned} \dim \mathbb{C}[D]^\Gamma &= \sum_{\gamma \in I} \langle \mathrm{inv}_D(e^\gamma), e^\gamma \rangle \\ &= |I| \left\{ \epsilon \left(\frac{-1}{p}\right)^{n/2} \frac{1}{p^2-1}\, \frac{1}{p^{(n-2)/2}}\, (p-1) \right\} \frac{1}{p^2-1} \left\{ |I \backslash \{0\}| + (p-1) \right\} \\ &= \frac{p^{n-1} - p}{p^2 - 1} + \epsilon \left(\frac{-1}{p}\right)^{n/2} p^{(n-2)/2} + 1 \end{aligned}$$

by Proposition 1.1.1. □

We describe an example. If D is of type $p^{\epsilon 2}$ with $\epsilon = \left(\frac{-1}{p}\right)$, the subspace of invariants has dimension $\dim \mathbb{C}[D]^\Gamma = 2$. The discriminant form D is generated by two isotropic elements γ_1, γ_2 such that $(\gamma_1, \gamma_2) = 1/p \mod 1$. We have

$$\mathrm{inv}_D(e^0) = \frac{1}{p+1}\left\{ e^0 + \sum_{\mu \in I} e^\mu \right\}$$

and

$$\mathrm{inv}_D(e^{\gamma_1}) = \frac{1}{p-1}\sum_{\mu\in\langle\gamma_1\rangle} e^\mu - \frac{1}{p^2-1}\left\{ e^0 + \sum_{\mu \in I} e^\mu \right\}.$$

This implies that $\mathbb{C}[D]^\Gamma$ is generated by the elements $\sum_{\mu\in\langle\gamma_1\rangle} e^\mu$ and $\sum_{\mu\in\langle\gamma_2\rangle} e^\mu$, which is a special case of Skoruppa's result.

As for odd primes we can prove

Theorem 3.2.4. *Let D be a discriminant form of type $2_{II}^{\epsilon n}$ with n even. Then*

$$\dim \mathbb{C}[D]^\Gamma = \frac{2^{n-1}+1}{3} + \epsilon\, 2^{(n-2)/2}\,.$$

The dimension formulas in Theorems 3.2.3 and 3.2.4 have also been found by Zemel using a slightly different approach (see Theorem 5.6 in [73]). We also remark that the numerical values of $\dim(\mathbb{C}[D]^\Gamma)$ for some of the above discriminant forms and others have been determined by Skoruppa and Ehlen (see Section 6 in [24]).

Corollary 3.2.5. *Let p be an odd prime and D a discriminant form of type $p^{\epsilon 3}$ with $\epsilon = \pm 1$. Choose $\gamma \in I\backslash\{0\}$. For $j \in \mathbb{Z}/p\mathbb{Z}$ define*

$$M(\gamma)_j = \{\, \mu \in I\backslash\{0\} \mid (\mu,\gamma) = j/p \mod 1 \,\}\,.$$

Let

$$M(\gamma)^+ = \bigcup_{\substack{j\in(\mathbb{Z}/p\mathbb{Z})^\times \\ \varepsilon\chi_D(j)=+1}} M(\gamma)_j \cup \bigcup_{\substack{j\in(\mathbb{Z}/p\mathbb{Z})^\times \\ \chi_D(j)=+1}} \{j\gamma\}$$

with $\varepsilon = \epsilon\left(\frac{2}{p}\right)$ and analogously $M(\gamma)^-$. Then $\mathbb{C}[D]^\Gamma$ is 1-dimensional and spanned by

$$\sum_{\mu\in M(\gamma)^+} e^\mu - \sum_{\mu\in M(\gamma)^-} e^\mu\,.$$

If D is of type p^{-4} where p is an odd prime or of type 2_{II}^{-4}, then $\mathbb{C}[D]^\Gamma$ is 1-dimensional and spanned by

$$pe^0 - \sum_{\mu\in I} e^\mu$$

with $p = 2$ in the latter case.

Proof. In the first case $\mathbb{C}[D]^\Gamma$ is spanned by $\mathrm{inv}_D(e^\gamma)$ for any $\gamma \in I\backslash\{0\}$ and in the second case by $\mathrm{inv}_D(e^0)$. □

The decomposition $I\backslash\{0\} = M(\gamma)^+ \cup M(\gamma)^-$ is independent of the choice of $\gamma \in I\backslash\{0\}$ and is equal to the decomposition of $I\backslash\{0\}$ under the action of the spinor kernel of $\mathrm{SO}(D)$. The size of $M(\gamma)^\pm$ is $(p^2-1)/2$.

3.3 Some 2-adic exercises

We study some 2-adic discriminant forms which will play an important role in our main theorem on invariants.

Let D be a discriminant form of type $2_t^{\epsilon n}$. Then D^{2*} contains a single element, which we denote by x_2. The signature of D is even if and only if n is even. In this case the matrix $Z = -1 \in \Gamma$ acts as multiplication by $e(-t/4) = e(t/4)$ so that there are no non-trivial invariants if $t = 2 \mod 4$.

Proposition 3.3.1. *Let D be a discriminant form of type $2_t^{\epsilon n}$ with n even and $t = 0 \mod 4$. Then*

$$\mathrm{inv}_D(e^\gamma) = \frac{1}{6}e^\gamma + \frac{1}{6}e^{\gamma+x_2} + \epsilon\,(-1)^{t/4}\,\frac{1}{6}\,\frac{1}{2^{(n-4)/2}}\left\{\sum_{\mu\in(\gamma^\perp\cap I)} 2\,e^\mu - \sum_{\mu\in I} e^\mu\right\}$$

for $\gamma \in I$ and

$$\dim \mathbb{C}[D]^\Gamma = \frac{2^{n-3}+1}{3} + \epsilon(-1)^{t/4}2^{(n-4)/2}\,.$$

The proof is similar to the proof of the next theorem. We therefore omit it. We describe two examples. If D is of type 2_0^{+2}, then $\mathbb{C}[D]^\Gamma$ is 1-dimensional and spanned by $\mathrm{inv}_D(e^0) = \mathrm{inv}_D(e^{x_2}) = (e^0 + e^{x_2})/2$. If D is of type $2_0^{-4} \cong 2_4^{+4}$, then $\mathbb{C}[D]^\Gamma$ is trivial.

Let D be a discriminant form of type $2_t^{\epsilon n}4_{II}^{+2}$. Then D has even signature if and only if n is even.

Proposition 3.3.2. *Let D be a discriminant form of type of type $2_t^{\epsilon n}4_{II}^{+2}$ with n even. Then*

$$\begin{aligned}\mathrm{inv}_D(e^\gamma) = \frac{1}{12}\{e^\gamma + e(t/4)e^{-\gamma}\} \\ + \frac{1}{24}\sum_{\mu\in(\gamma+D^{2*})\cap I} e(\mathrm{q}_2(\mu-\gamma))\{e^\mu + e(t/4)e^{-\mu}\} \\ + \epsilon\, e(5t/8)\,\frac{1}{12}\,\frac{1}{2^{n/2}}\sum_{\mu\in I} e((\mu,\gamma))\{e^\mu + e(t/4)e^{-\mu}\}\end{aligned}$$

for $\gamma \in I$ and

$$\dim \mathbb{C}[D]^\Gamma = \frac{1}{12}\,|I|\big\{1+\epsilon\, e(5t/8)\frac{1}{2^{n/2}}(1+e(t/4))\big\} + \frac{1}{12}\,e(t/4)\,|I_2|$$

with

$$|I| = 2^{n+2} + \epsilon\, 2^{(n+2)/2}\,\delta(t/4)\left(\frac{t-1}{2}\right)$$
$$|I_2| = 2^{n} + \epsilon\, 2^{(n+2)/2}\,\delta(t/4)\left(\frac{t-1}{2}\right).$$

Proof. First note that $e(\mathrm{sign}(D)/8) = \gamma_2(D) = \epsilon\, e(t/8)$. The group $\Gamma(4)$ has 6 cusps s, which can be represented by $1/4, 1/2$ and $a/1$ with $a = 0,1,2,3$. Choosing matrices M_s as $\left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}a&-1\\1&0\end{smallmatrix}\right)$ we find

$$\mathrm{inv}_D(e^\gamma)_{1/4} = \frac{1}{12}\{e^\gamma + e(t/4)e^{-\gamma}\}$$

and

$$\mathrm{inv}_D(e^\gamma)_{1/2} = \frac{1}{24}\sum_{\mu\in(\gamma+D^{2*})\cap I} e(-\,\mathrm{q}_2(\mu-\gamma))\{e^\mu + e(t/4)e^{-\mu}\}\,.$$

We remark that in the last sum $e(-\,\mathrm{q}_2(\mu-\gamma)) = e(\mathrm{q}_2(\mu-\gamma)) = \pm 1$. For $M_s = \left(\begin{smallmatrix}a&-1\\1&0\end{smallmatrix}\right)$ we have $\xi(M_s^{-1}) = \epsilon\, e(5t/8)$ and $a\gamma + D^{1*} = D$ so that

$$\mathrm{inv}_D(e^\gamma)_{a/1} = \epsilon\, e(5t/8)\,\frac{1}{48}\,\frac{1}{2^{n/2}}\sum_{\mu\in I} e((\mu,\gamma))\{e^\mu + e(t/4)e^{-\mu}\}\,.$$

This implies the formula for $\mathrm{inv}_D(e^\gamma)$.
Next we calculate the dimension of the fixed point subspace. For $\gamma \in I$ we have

$$\langle \mathrm{inv}_D(e^\gamma)_{1/4}, e^\gamma\rangle = \frac{1}{12} + \frac{1}{12}\,e(t/4)\begin{cases}1 & \text{if } 2\gamma = 0,\\ 0 & \text{otherwise}\end{cases}$$
$$\langle \mathrm{inv}_D(e^\gamma)_{1/2}, e^\gamma\rangle = 0$$
$$\langle \mathrm{inv}_D(e^\gamma)_{a/1}, e^\gamma\rangle = \epsilon\, e(5t/8)\,\frac{1}{48}\,\frac{1}{2^{n/2}}(1+e(t/4))$$

so that

$$\begin{aligned}\dim \mathbb{C}[D]^\Gamma &= \sum_{\gamma\in I}\langle \mathrm{inv}_D(e^\gamma), e^\gamma\rangle \\ &= \frac{1}{12}\,|I|\big\{1+\epsilon\, e(5t/8)\,\frac{1}{2^{n/2}}(1+e(t/4))\big\} + \frac{1}{12}\,e(t/4)\,|I_2|,\end{aligned}$$

where $I_2 = I \cap D_2$. The cardinalities of I and I_2 can be determined with Proposition 1.1.3. □

Note that if $t = 2 \mod 4$ and $2\gamma = 0$, then $\mathrm{inv}_D(e^\gamma) = 0$. This also follows from the formula for the action of $Z = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Now we consider the case that D is of type $2_t^{+2} 4_{II}^{+2}$ with $t = 2 \mod 4$. Then $\mathrm{sign}(D) = t \bmod 8$. The set $I \backslash I_2$ has cardinality $16 - 4 = 12$ and $\mathrm{O}(D)$ acts transitively on it. Let $\gamma \in I \backslash I_2$. For $j \in \mathbb{Z}/4\mathbb{Z}$ we define

$$M(\gamma)_j = \{ \mu \in I \backslash I_2 \,|\, (\mu, \gamma) = j/4 \mod 1 \} .$$

Then

$$|M(\gamma)_j| = \begin{cases} 4 & \text{if } j \text{ is odd,} \\ 2 & \text{if } j \text{ is even.} \end{cases}$$

We have $M(\gamma)_0 = \{\pm\gamma\}$. There is a unique element $\mu \in D^{2*}$ such that $\mathrm{q}_2(\mu) = 0 \mod 1$ and $(\mu, \gamma) = 1/2 \mod 1$. Define $\alpha = \mu + \gamma$. Then $M(\gamma)_2 = \{\pm\alpha\}$. Let

$$M(\gamma)^+ = M(\gamma)_j \cup \{+\alpha\} \cup \{+\gamma\}$$

with $j \in (\mathbb{Z}/4\mathbb{Z})^\times$ such that $\varepsilon\chi_d(j) = +1$ and

$$M(\gamma)^- = M(\gamma)_j \cup \{-\alpha\} \cup \{-\gamma\}$$

with $j \in (\mathbb{Z}/4\mathbb{Z})^\times$ such that $\varepsilon\chi_D(j) = -1$, where in both cases

$$\varepsilon = \begin{cases} 1 & \text{if } t = 6 \mod 8 \,, \\ -1 & \text{if } t = 2 \mod 8 \,. \end{cases}$$

Proposition 3.3.3. *Let D be a discriminant form of type $2_t^{+2} 4_{II}^{+2}$ with $t = 2 \bmod 4$. Then the subspace of invariants $\mathbb{C}[D]^\Gamma$ is 1-dimensional and spanned by*

$$\sum_{\mu \in M(\gamma)^+} e^\mu - \sum_{\mu \in M(\gamma)^-} e^\mu ,$$

where γ is any element in $I \backslash I_2$.

Proof. By the previous proposition

$$\dim \mathbb{C}[D]^\Gamma = \frac{1}{12}(|I| - |I_2|) = \frac{1}{12}(16 - 4) = 1 \,.$$

For $\gamma \in I \backslash I_2$ we have

$$\begin{aligned} \mathrm{inv}_D(e^\gamma) = \frac{1}{12}\{e^\gamma - e^{-\gamma}\} & \\ + \frac{1}{24} \sum_{\mu \in (\gamma + D^{2*}) \cap I} e(\mathrm{q}_2(\mu - \gamma))\{e^\mu - e^{-\mu}\} & \\ + e(5t/8)\, \frac{1}{24} \sum_{\substack{\mu \in I \backslash I_2 \\ (\mu,\gamma) = \pm 1/4}} e((\mu, \gamma))\{e^\mu - e^{-\mu}\} \,. & \end{aligned}$$

The sum is supported on $I\backslash I_2$ (see Proposition 3.1.6). We easily check that

$$\operatorname{inv}_D(e^\gamma) = \frac{1}{12}\left\{ \sum_{\mu\in M(\gamma)^+} e^\mu - \sum_{\mu\in M(\gamma)^-} e^\mu \right\}.$$

This proves the proposition. □

The decomposition $I\backslash I_2 = M(\gamma)^+ \cup M(\gamma)^-$ is independent of the choice of $\gamma \in I\backslash I_2$. The size of $M(\gamma)^\pm$ is $4+1+1 = 6 = 12/2$.

We remark that every discriminant form D of level 4, exponent 4, order 4^3 and signature $t = 2 \mod 4$ is isomorphic to $2_t^{+2}4_{II}^{+2}$.

Next we consider a discriminant form D of type of type $2_1^{+1}4_t^\epsilon 8_{II}^{+2}$ with $t = 1 \mod 2$ and $\epsilon = \left(\frac{t}{2}\right)$. Then $\operatorname{sign}(D) = 1 + t \mod 8$. Recall that $I_4 = I \cap D_4$.

Proposition 3.3.4. *We have $|I| = 64$ and $|I_4| = 16$.*

Proof. The partition function of 8_{II}^{+2} is given by

$$f_{8_{II}^{+2}}(x) = \sum_{\gamma\in 8_{II}^{+2}} x^{8\mathrm{q}(\gamma)} = 20 + 4(x + x^3 + x^5 + x^7) + 8(x^2 + x^6) + 12x^4,$$

where we have chosen $\mathrm{q}(\gamma) \in [0, 1)$. Multiplying this polynomial with the polynomials $f_{2_1^{+1}}(x) = 1 + x^2$ and $f_{4_t^\epsilon}(x) = 1 + 2x^t + x^4$ we can easily derive the first statement. The second follows analogously. □

Proposition 3.3.5. *The group $\mathrm{O}(D)$ acts transitively on $I\backslash I_4$.*

Proof. Let $\gamma \in I\backslash I_4$. Then there is an element $\beta \in D$ such that $(\gamma, \beta) = 1/8 \mod 1$. Define $\mu = \beta - a\gamma$, where $a = 8\,\mathrm{q}(\beta) \mod 8$. Then $\langle\gamma, \mu\rangle$ is a discriminant form of type 8_{II}^{+2}. The orthogonal complement $\langle\gamma, \mu\rangle^\perp$ is a discriminant form of type $2_{t_2}^{\epsilon_2}4_{t_4}^{\epsilon_4}$. Up to isomorphism there are exactly 4 such forms namely the forms of type $2_1^{+1}4_{t_4}^{\epsilon_4}$ with t_4 odd and $\epsilon_4 = \left(\frac{t_4}{2}\right)$. The signature of such a form is $1 + t_4 \mod 8$. Hence, each element γ in $I\backslash I_4$ gives rise to a Jordan decomposition $2_1^{+1}4_t^\epsilon 8_{II}^{+2}$. This implies that all elements in $I\backslash I_4$ are conjugate under $\mathrm{O}(D)$. □

Proposition 3.3.6. *Let $\gamma \in I_4$. Then $\operatorname{inv}_D(e^\gamma) = 0$.*

Proof. We have $4\gamma = 0 \in I^\perp$ so that $\operatorname{inv}_D(e^\gamma) = 0$ by Proposition 3.1.7. □

Proposition 3.3.7. *For $\gamma \in I$ we have*

$$\begin{aligned}
\operatorname{inv}_D(e^\gamma) = {} & e(\operatorname{sign}(D)/8)\, \frac{1}{48}\, \frac{1}{2\sqrt{2}} \sum_{\mu \in I} e((\mu,\gamma))(1 - e(4(\mu,\gamma))) \\
& \qquad \{e^\mu + e(\operatorname{sign}(D)/4) e^{-\mu}\} \\
& + \epsilon\, e(t/8)\, \frac{1}{48}\, \frac{1}{4\sqrt{2}} \sum_{\substack{a \in \mathbb{Z}/8\mathbb{Z} \\ a = 1 \bmod 2}} \sum_{\mu \in (a\gamma + D^{2*}) \cap I} e(-\operatorname{q}_2(\mu - a\gamma))\, e(\tfrac{1-a}{2}(\mu,\gamma)) \\
& \qquad \{e^\mu + e(\operatorname{sign}(D)/4) e^{-\mu}\} \\
& + \frac{1}{48}\, \frac{1}{2} \sum_{\substack{a \in \mathbb{Z}/8\mathbb{Z} \\ a = 1 \bmod 4}} \sum_{\mu \in (a\gamma + D^{4*}) \cap I} e(-\operatorname{q}_4(\mu - a\gamma))\, e(\tfrac{(1-a)}{4}(\mu,\gamma)) \\
& \qquad \{e^\mu + e(\operatorname{sign}(D)/4) e^{-\mu}\} \\
& + \frac{1}{48} \sum_{\substack{a \in \mathbb{Z}/8\mathbb{Z} \\ a = 1 \bmod 2}} \chi_D(a)\, e^{a\gamma} .
\end{aligned}$$

Proof. The group $\Gamma(8)$ has 24 cusps.
There are 16 cusps $s = a/c \in \mathbb{Q}$, $(a,c) = 1$ with $(c,8) = 1$. For such a cusp we can choose a matrix $M_s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ with $d = 0 \bmod 8$. Then $b = -c \bmod 8$ and $\xi(M_s^{-1}) = \left(\frac{c}{2}\right) e(c\operatorname{sign}(D)/8)$ so that

$$\operatorname{inv}_D(e^\gamma)_s = \left(\frac{c}{2}\right) e(c\operatorname{sign}(D)/8)\, \frac{1}{48}\, \frac{1}{16\sqrt{2}} \sum_{\mu \in I} e(c(\mu,\gamma)) \{e^\mu + e(\operatorname{sign}(D)/4) e^{-\mu}\}.$$

There are 4 cusps $s = a/c \in \mathbb{Q}$, $(a,c) = 1$ with $(c,8) = 2$. We can choose a matrix $M_s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ such that $d = 1 \bmod 16$. Then $b = c(a-1)/4 \bmod 8$ and $\xi(M_s^{-1}) = \epsilon\, e(t/8)$. It follows

$$\begin{aligned}
\operatorname{inv}_D(e^\gamma)_s = {} & \epsilon\, e(t/8)\, \frac{1}{48}\, \frac{1}{4\sqrt{2}} \\
& \sum_{\mu \in (a\gamma + D^{2*}) \cap I} e(-\tfrac{c}{2} \operatorname{q}_2(\mu - a\gamma))\, e(\tfrac{c(1-a)}{4}(\mu,\gamma))\, \{e^\mu + e(\operatorname{sign}(D)/4) e^{-\mu}\} .
\end{aligned}$$

There are 2 cusps $s = a/c \in \mathbb{Q}$, $(a,c) = 1$ with $(c,8) = 4$. We choose a representative $s = a/c$ with $a = 1 \bmod 4$. Then there is a matrix $M_s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ such that $d = 1 \bmod 32$. We find $b = c(a-1)/16 \bmod 8$ and $\xi(M_s^{-1}) = 1$ so that

$$\begin{aligned}
\operatorname{inv}_D(e^\gamma)_s = {} & \frac{1}{48}\, \frac{1}{2} \\
& \sum_{\mu \in (a\gamma + D^{4*}) \cap I} e(-\tfrac{c}{4} \operatorname{q}_4(\mu - a\gamma))\, e(\tfrac{c(1-a)}{16}(\mu,\gamma))\, \{e^\mu + e(\operatorname{sign}(D)/4) e^{-\mu}\} .
\end{aligned}$$

Finally, there are 2 cusps $s = a/c \in \mathbb{Q}$, $(a, c) = 1$ with $(c, 8) = 8$. We choose a matrix $M_s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Then $a = d \mod 8$ and $\xi(M_s^{-1}) = \left(\frac{a}{2}\right) e((a-1)\,\mathrm{sign}(D)/8)$ so that

$$\mathrm{inv}_D(e^\gamma)_s = \left(\frac{a}{2}\right) e((a-1)\,\mathrm{sign}(D)/8)\,\frac{1}{48}\,\{e^{a\gamma} + e(\mathrm{sign}(D)/4)e^{-a\gamma}\}\,.$$

Putting the contributions of the cusps together we obtain the given formula. □

Proposition 3.3.8. *Let D be a discriminant form of type $2_1^{+1}4_t^{\epsilon}\,8_{II}^{+2}$ with $t = 1 \mod 2$ and $\epsilon = \left(\frac{t}{2}\right)$. Then $\mathbb{C}[D]^\Gamma$ is 1-dimensional.*

Proof. We have

$$\dim \mathbb{C}[D]^\Gamma = \sum_{\gamma \in I} \langle \mathrm{inv}_D(e^\gamma), e^\gamma \rangle = \sum_{\gamma \in I \backslash I_4} \langle \mathrm{inv}_D(e^\gamma), e^\gamma \rangle$$
$$= \sum_{\gamma \in I \backslash I_4} \sum_{s \in \Gamma(N) \backslash P} \langle \mathrm{inv}_D(e^\gamma)_s, e^\gamma \rangle.$$

by Proposition 3.3.6. The cusps $s = a/c$ with $(c, 8) = 1$ do not contribute to the last sum because $1 - e(4(\mu, \gamma)) = 0$ for $\mu = \pm\gamma$. Furthermore, for $\gamma \in I$ we have $\pm\gamma \notin a\gamma + D^{2*}$ because $\mathrm{q}(x_2) = 1/4 \mod 1$ and analogously $\pm\gamma \notin a\gamma + D^{4*}$ because $\mathrm{q}(x_4) = 1/2 \mod 1$. Hence, the only contribution to the last sum comes from the cusp $1/8$. It follows

$$\dim \mathbb{C}[D]^\Gamma = \frac{1}{48} \sum_{\gamma \in I \backslash I_4} 1 = \frac{1}{48}(|I| - |I_4|) = \frac{1}{48}(64 - 16) = 1\,.$$

This proves the proposition. □

Finally, we show that the generator of $\mathbb{C}[D]^\Gamma$ can be written analogously to the cases $p^{\epsilon 3}$ and $2_t^{+2}4_{II}^{+2}$ (see Corollary 3.2.5 and Proposition 3.3.3). Fix a Jordan decomposition $2_1^{+1}4_t^{\epsilon}\,8_{II}^{+2}$ with $t = 1 \mod 2$ and $\epsilon = \left(\frac{t}{2}\right)$ of D. Let $\gamma \in I \backslash I_4$. For $j \in \mathbb{Z}/8\mathbb{Z}$ we define

$$M(\gamma)_j = \{\, \mu \in I \backslash I_4 \,|\, (\mu, \gamma) = j/8 \mod 1 \,\}\,.$$

Then $aM(\gamma)_j = M(\gamma)_{aj}$ for all $a \in (\mathbb{Z}/8\mathbb{Z})^\times$ and

$$|M(\gamma)_j| = \begin{cases} 8 & \text{if } j \text{ is odd,} \\ 4 & \text{if } j \text{ is even.} \end{cases}$$

We describe the sets $M(\gamma)_j$ explicitly for even j. We have

$$M(\gamma)_0 = \{ j\gamma \,|\, j \in (\mathbb{Z}/8\mathbb{Z})^\times \}\,.$$

There is a unique element $\mu \in D^{4*}$ such that $\mathrm{q}_4(\mu) = 0 \mod 1$ and $(\mu, \gamma) = 1/2 \mod 1$. Define $\alpha_4 = \mu + \gamma$. Then

$$M(\gamma)_4 = \{ j\alpha_4 \,|\, j \in (\mathbb{Z}/8\mathbb{Z})^\times \} \,.$$

Finally, there are exactly two elements $\mu_i \in D^{2*}$, $i = 1, 2$ such that $\mathrm{q}_2(\mu_i) = t/4 \mod 1$ and $(\mu_i, \gamma) = 1/4 \mod 1$. Define $\alpha_i = \mu_i + \gamma$. Then

$$M(\gamma)_2 = \{\alpha_1, 5\alpha_1, \alpha_2, 5\alpha_2\} \quad \text{and} \quad M(\gamma)_6 = \{3\alpha_1, 7\alpha_1, 3\alpha_2, 7\alpha_2\} \,.$$

These statements can be proved by choosing generators of $2_1^{+1} 4_t^\epsilon \, 8_{II}^{+2}$ and assuming that γ is one of the two isotropic generators of 8_{II}^{+2}. (Recall that by Proposition 3.3.5 the elements in $I \backslash I_4$ are conjugate under $\mathrm{O}(D)$.) We decompose $I \backslash I_4 = M(\gamma)^+ \cup M(\gamma)^-$ with

$$M(\gamma)^+ = \bigcup_{\substack{j \in (\mathbb{Z}/8\mathbb{Z})^\times \\ \varepsilon\chi_D(j) = +1}} M(\gamma)_j \cup \bigcup_{\substack{j \in (\mathbb{Z}/8\mathbb{Z})^\times \\ \chi_D(j) = +1}} \{j\alpha_1, j\alpha_2, j\alpha_4, j\gamma\}$$

and

$$M(\gamma)^- = \bigcup_{\substack{j \in (\mathbb{Z}/8\mathbb{Z})^\times \\ \varepsilon\chi_D(j) = -1}} M(\gamma)_j \cup \bigcup_{\substack{j \in (\mathbb{Z}/8\mathbb{Z})^\times \\ \chi_D(j) = -1}} \{j\alpha_1, j\alpha_2, j\alpha_4, j\gamma\},$$

where

$$\varepsilon = \begin{cases} 1 & \text{if } t = 5 \text{ or } 7 \mod 8 \,, \\ -1 & \text{if } t = 1 \text{ or } 3 \mod 8 \,. \end{cases}$$

Proposition 3.3.9. *Let D be a discriminant form of type $2_1^{+1} 4_t^\epsilon \, 8_{II}^{+2}$ with $t = 1 \mod 2$ and $\epsilon = \left(\frac{t}{2}\right)$. Then $\mathbb{C}[D]^\Gamma$ is spanned by*

$$\sum_{\mu \in M(\gamma)^+} e^\mu - \sum_{\mu \in M(\gamma)^-} e^\mu,$$

where γ is any element in $I \backslash I_4$.

Proof. Let $\gamma \in I \backslash I_4$. We write $\mathrm{inv}_D(e^\gamma) = \sum_{\mu \in I} c_\mu e^\mu$. Then $c_\mu = 0$ for $\mu \in I_4$ by Proposition 3.3.6. Now we consider the individual sums in Proposition 3.3.7. The first sum extends over $\bigcup_{j \in (\mathbb{Z}/8\mathbb{Z})^\times} M(\gamma)_j$, the second over $M(\gamma)_2 \cup M(\gamma)_6$, the third over $M(\gamma)_4$ and the last sum over $M(\gamma)_0$. For the first sum we find

$$\begin{aligned} &\sum_{\mu \in I \backslash I_4} e((\mu, \gamma))(1 - e(4(\mu, \gamma)))\{e^\mu + e(\mathrm{sign}(D)/4)e^{-\mu}\} \\ &\qquad = 2 \sum_{j \in (\mathbb{Z}/8\mathbb{Z})^\times} \sum_{\mu \in M(\gamma)_j} \{e(j/8) + e(\mathrm{sign}(D)/4)e(-j/8)\} e^\mu \\ &\qquad = 2\sqrt{2} \sum_{j \in (\mathbb{Z}/8\mathbb{Z})^\times} \chi_D(j) \sum_{\mu \in M(\gamma)_j} e^\mu \begin{cases} 1 & \text{if } t = 3 \mod 4, \\ i & \text{if } t = 1 \mod 4 \end{cases} \end{aligned}$$

so that

$$e(\operatorname{sign}(D)/8)\,\frac{1}{48}\,\frac{1}{2\sqrt{2}}\sum_{\mu\in I\backslash I_4} e((\mu,\gamma))(1-e(4(\mu,\gamma)))\{e^{\mu}+e(\operatorname{sign}(D)/4)e^{-\mu}\}$$
$$= \varepsilon\,\frac{1}{48}\sum_{j\in(\mathbb{Z}/8\mathbb{Z})^{\times}}\chi_D(j)\sum_{\mu\in M(\gamma)_j} e^{\mu}\,.$$

We calculate the second sum as

$$\begin{aligned}
&\epsilon\, e(t/8)\sum_{\substack{a\in\mathbb{Z}/8\mathbb{Z}\\ a=1 \bmod 2}}\ \sum_{\mu\in(a\gamma+D^{2*})\cap I} e(-\operatorname{q}_2(\mu-a\gamma))\, e(\tfrac{1-a}{2}(\mu,\gamma))\\
&\qquad\{e^{\mu}+e(\operatorname{sign}(D)/4)e^{-\mu}\}\\
&= \epsilon\, e(t/8)\sum_{\mu\in M(\gamma)_2}\ \sum_{a\in(\mathbb{Z}/8\mathbb{Z})^{\times}}\{e((1-a)/8)e(-\operatorname{q}_2(\mu-a\gamma))+\\
&\qquad e(\operatorname{sign}(D)/4)e(3(1-a)/8)e(-\operatorname{q}_2(-\mu-a\gamma))\}e^{\mu}\\
&\quad+\epsilon\, e(t/8)\sum_{\mu\in M(\gamma)_6}\ \sum_{a\in(\mathbb{Z}/8\mathbb{Z})^{\times}}\{e(3(1-a)/8)e(-\operatorname{q}_2(\mu-a\gamma))+\\
&\qquad e(\operatorname{sign}(D)/4)e((1-a)/8)e(-\operatorname{q}_2(-\mu-a\gamma))\}e^{\mu}\\
&= 4\sqrt{2}\left\{\sum_{j\in(\mathbb{Z}/8\mathbb{Z})^{\times}}\chi_D(j)\,e^{j\alpha_1}+\sum_{j\in(\mathbb{Z}/8\mathbb{Z})^{\times}}\chi_D(j)\,e^{j\alpha_2}\right\}
\end{aligned}$$

Finally, we consider the third sum. We easily see that

$$\sum_{\substack{a\in\mathbb{Z}/8\mathbb{Z}\\ a=1 \bmod 4}}\ \sum_{\mu\in(a\gamma+D^{4*})\cap(I\backslash I_4)} e(-\operatorname{q}_4(\mu-\gamma))\, e(\tfrac{(1-a)}{4}(\mu,\gamma))\{e^{\mu}+e(\operatorname{sign}(D)/4)e^{-\mu}\}$$
$$= 2\sum_{j\in(\mathbb{Z}/8\mathbb{Z})^{\times}}\chi_D(j)\,e^{j\alpha}\,.$$

Putting these contributions together we get

$$\operatorname{inv}_D(e^{\gamma}) = \frac{1}{48}\left\{\sum_{\mu\in M(\gamma)^+} e^{\mu}-\sum_{\mu\in M(\gamma)^-} e^{\mu}\right\}.$$

This proves the proposition. □

Note that the decomposition $I\backslash I_4 = M(\gamma)^+\cup M(\gamma)^-$ is independent of the choice of γ because $\mathbb{C}[D]^{\Gamma}$ is 1-dimensional. The sets $M(\gamma)^{\pm}$ have size $2\cdot 8+2\cdot 4 = 24 = 48/2$.

We remark that every discriminant form D of level 8, exponent 8, order 8^3 and even signature $1+t \bmod 8$ is isomorphic to $2_1^{+1}4_t^{\epsilon}\,8_{II}^{+2}$ with $\epsilon = \left(\frac{t}{2}\right)$.

3.4 Main theorem on invariants

In this section we prove the main result of this chapter. We define fundamental invariants and show that each invariant is induced from these invariants.

As explained in the introduction we can restrict to p-adic discriminant forms. We take another look at the isotropic lifts and show some results similar to those in section 2.3 relevant for invariants. Let D be a discriminant form of level p^l, where p is a prime and even signature. For $\gamma \in D$ we define

$$a(p,\gamma) = |\{H \subset \gamma^\perp \text{ is an isotropic subgroup of } D \text{ with } |H| = p\}| .$$

Recall Lemma 2.3.1 that gave a sufficient condition for an element $e^\gamma \in \mathbb{C}[D]$ to be a linear combination of isotropic lifts. We now have

Proposition 3.4.1. *Let $\gamma \in I\backslash\{0\}$. Then $\gamma^\perp$ contains an isotropic subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$ if and only if $a(p,\gamma) > 1$.*

Proof. Let γ be of order n. Then $(n/p)\gamma$ generates an isotropic subgroup of order p in $\gamma^\perp$. Since $a(p,\gamma) > 1$, there is another isotropic subgroup of order p in $\gamma^\perp$. Both groups together generate an isotropic subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$ in $\gamma^\perp$. $\square$

Now we have to distinguish between even and odd primes.

Proposition 3.4.2. *Let D be a discriminant form of level p^l, where p is an odd prime. Let $\gamma \in I$ be of order p. If $a(p,\gamma) = 1$, then $\mathrm{inv}_D(e^\gamma) = \mathrm{inv}_D(e^0)$ or D is of type $p^{\epsilon 2}$ with $\epsilon = \left(\frac{-1}{p}\right)$, $p^{\pm 3}$ or p^{-4}.*

Proof. First we consider the case $\gamma \notin D^p$. We show that $\langle\gamma\rangle^\perp/\langle\gamma\rangle$ contains no nontrivial isotropic elements. Suppose $\mu + \langle\gamma\rangle \in \langle\gamma\rangle^\perp/\langle\gamma\rangle$ with $\mu \notin \langle\gamma\rangle$ is isotropic. Then $\mu \in \langle\gamma\rangle^\perp$ is isotropic and $a(p,\gamma) = 1$ implies $(n/p)\mu \in \langle\gamma\rangle$, where n is the order of μ. Since $\gamma \notin D^p$, we conclude $\mu \in \langle\gamma\rangle$. It follows that $\langle\gamma\rangle^\perp/\langle\gamma\rangle$ is of type 0, $p^{\pm 1}$ or $p^{-\epsilon 2}$ with $\epsilon = \left(\frac{-1}{p}\right)$. If $\langle\gamma\rangle^\perp/\langle\gamma\rangle = 0$, then $|D| = p^2$ so that D is isomorphic to $q^{\pm 1}$ with $q = p^2$ or to $p^{\epsilon 2}$. The first case contradicts $\gamma \notin D^p$. Hence, D is isomorphic to $p^{\epsilon 2}$. If $\langle\gamma\rangle^\perp/\langle\gamma\rangle = p^{\pm 1}$, then $|D| = p^3$ and D must be of type $p^{\pm 3}$. For $\langle\gamma\rangle^\perp/\langle\gamma\rangle \cong p^{-\epsilon 2}$ we find $D \cong p^{-4}$.

Now let $\gamma \in D^p$. We choose a Jordan decomposition of D and write $D = A \oplus B$, where A denotes the sum over the irreducible components of order p and $B \neq 0$ the sum over the remaining components. Then $\gamma \in B^p$. Recall that B^p is the orthogonal complement of B_p. Since B_p is isotropic and $a(p,\gamma) = 1$, we can conclude $B_p = \langle\gamma\rangle$, i.e. B_p is cyclic. This implies that B is cyclic. Let $B \cong q^{\pm 1}$. Then $\gamma = (q/p)\beta$ for some generator β of B. An isotropic element in D is of the form $\mu + m\beta$ with $\mu \in A$ and $p \,|\, m$. Since

$$(\gamma, \mu + m\beta) = (q/p)m(\beta, \beta) = 0 \mod 1,$$

this implies $\gamma \in I^{\perp}$. Hence, $\mathrm{inv}_D(e^{\gamma}) = \mathrm{inv}_D(e^0)$ by Proposition 3.1.5. □

We continue with the case $p = 2$. We will use Lemma 2.3.13 in the following three propositions.

Proposition 3.4.3. *Let D be a discriminant form of level 2^l such that χ_D is trivial. Let $\gamma \in I$ be of order 2. If $a(2, \gamma) = 1$, then $\mathrm{inv}_D(e^{\gamma}) = \mathrm{inv}_D(e^0)$ or D is of type 2_{II}^{+2} or 2_{II}^{-4}.*

Proof. Note that the condition on χ_D implies that $|D|$ is a square and $\mathrm{sign}(D) = 0 \mod 4$.
First we consider the case $\gamma \notin D^2$. The discriminant form $\langle\gamma\rangle^{\perp}/\langle\gamma\rangle$ has the same signature and square class as D and contains no non-trivial isotropic elements. Hence, $\langle\gamma\rangle^{\perp}/\langle\gamma\rangle$ is isomorphic to 0 or 2_{II}^{-2}. If $\langle\gamma\rangle^{\perp}/\langle\gamma\rangle \cong 0$, then $|D| = 2^2$ and D contains a non-trivial isotropic element of order 2. This implies $D \cong 2_{II}^{+2}$ or $D \cong 2_0^{+2}$. In the latter case $\mathbb{C}[D]^{\Gamma}$ is spanned by $\mathrm{inv}_D(e^{\gamma}) = \mathrm{inv}_D(e^0)$ (cf. Proposition 3.3.1). If $\langle\gamma\rangle^{\perp}/\langle\gamma\rangle \cong 2_{II}^{-2}$, then D has order 16 and signature 4 mod 8. The discriminant forms of order 16 and signature 4 mod 8 are

$$4_4^{-2},\ 2_4^{+4},\ 2_{II}^{-4}.$$

In the first case the isotropic elements are multiples of 2. In the case 2_4^{+4} the space $\mathbb{C}[D]^{\Gamma}$ is trivial so that $\mathrm{inv}_D(e^{\gamma}) = \mathrm{inv}_D(e^0) = 0$ (cf. Proposition 3.3.1).
Next we assume that $\gamma \in D^2$. We choose a Jordan decomposition of D and write $D = A \oplus B$, where A denotes the sum over the irreducible components of exponent 2 and $B \neq 0$ the sum over the remaining components. Then $\gamma \in B^2$. The group B^2 is the orthogonal complement of $B_2 \subset B^2$, but in general B_2 is not isotropic. If B_2 is isotropic, we can argue exactly as in the proof of the previous proposition. Suppose B_2 is not isotropic. Since $a(2, \gamma) = 1$, the only non-trivial isotropic element in B_2 is γ. Hence, the discriminant form B must be of type $4_t^{\pm 2}$ or $4_s^{\pm 1} q_t^{\pm 1}$ with $8 \,|\, q$. In the latter case we can choose a generator β of $q_t^{\pm 1}$ such that $\gamma = (q/2)\beta$. Then $\gamma \in I^{\perp}$ so that $\mathrm{inv}_D(e^{\gamma}) = \mathrm{inv}_D(e^0)$ by Proposition 3.1.5. Suppose B is of type $4_t^{\pm 2}$. We choose orthogonal generators β_1, β_2 of B. Then $\gamma = 2\beta_1 + 2\beta_2$ and any isotropic element in D is of the form $\mu + m_1\beta_1 + m_2\beta_2$ with $\mu \in A$ and $2 \,|(m_1 + m_2)$. Now

$$(\gamma, \mu + m_1\beta_1 + m_2\beta_2) = 2m_1(\beta_1, \beta_1) + 2m_2(\beta_2, \beta_2) = 0 \mod 1,$$

implies $\gamma \in I^{\perp}$ so that again $\mathrm{inv}_D(e^{\gamma}) = \mathrm{inv}_D(e^0)$ by Proposition 3.1.5. □

Proposition 3.4.4. *Let D be a discriminant form of level 2^l and even signature such that χ_D is non-trivial and $|D|$ is a square. Let $\gamma \in I$ be of order 4. If $a(2,\gamma) = 1$, then $\mathrm{inv}_D(e^\gamma) = 0$ or D is of type $2_t^{+2}4_{II}^{+2}$ with $t = 2 \mod 4$.*

Proof. Since χ_D is non-trivial and $|D|$ is a square, we have $\mathrm{sign}(D) = 2 \mod 4$. First we consider the case $\gamma \notin D^2$. The discriminant form $\langle\gamma\rangle^\perp/\langle\gamma\rangle$ has the same signature and square class as D and contains no non-trivial isotropic elements. Hence, it is isomorphic to $2_t^{\pm 2}$ with $t = 2 \mod 4$. It follows that D has order 64. The discriminant forms of order 64 and signature 2 mod 4 containing elements of order 4 are

$$2_s^{\pm1}32_t^{\pm1},\ 4_s^{\pm1}16_t^{\pm1},\ 8_t^{\pm2},$$
$$2_s^{\pm3}8_t^{\pm1},\ 2_s^{\pm2}4_t^{\pm2},\ 2_{II}^{\pm2}4_t^{\pm2},$$
$$2_s^{\pm2}4_{II}^{\pm2}$$

with suitable s, t and signs. For the discriminant forms of type $2_s^{\pm1}32_t^{\pm1}$, $4_s^{\pm1}16_t^{\pm1}$ and $8_t^{\pm2}$ the isotropic elements of order 4 are multiples of 2. For the discriminant forms of type $2_s^{\pm3}8_t^{\pm1}$, $2_s^{\pm2}4_t^{\pm2}$ and $2_{II}^{\pm2}4_t^{\pm2}$ any isotropic element μ of order 4 satisfies $2\mu \in I^\perp$ so that $\mathrm{inv}_D(e^\mu) = 0$ by Proposition 3.1.6. Finally, $2_s^{\pm2}4_{II}^{\pm2} \cong 2_t^{+2}4_{II}^{+2}$ for some t with $t = 2 \mod 4$.
Now suppose $\gamma \in D^2$. As above, we choose a Jordan decomposition of D and write $D = A \oplus B$, where A denotes the sum over the irreducible components of exponent dividing 4 and $B \neq 0$ the sum over the remaining components. Then γ is orthogonal to B_2. Since B_2 is isotropic and $a(2,\gamma) = 1$, we have $B_2 = \langle 2\gamma\rangle$. Hence, B is cyclic. We can choose a generator β of $B \cong q_t^{\pm1}$ such that $\gamma = 2\alpha + (q/4)\beta$ for some $\alpha \in A$. An isotropic element in D is of the form $\mu + m\beta$ with $\mu \in A$ and $2\,|\,m$. Now

$$(2\gamma, \mu + m\beta) = (q/2)m(\beta,\beta) = 0 \mod 1$$

so that $2\gamma \in I^\perp$. Hence, $\mathrm{inv}_D(e^\gamma) = 0$ by Proposition 3.1.6. □

Proposition 3.4.5. *Let D be a discriminant form of level 2^l and even signature such that $|D|$ is not a square. Let $\gamma \in I$ be of order 8. If $a(2,\gamma) = 1$, then $\mathrm{inv}_D(e^\gamma) = 0$ or D is of type $2_1^{+1}4_t^{\epsilon}8_{II}^{+2}$ with $t = 1 \mod 2$ and $\epsilon = \left(\frac{t}{2}\right)$.*

Proof. As before we consider first the case that $\gamma \notin D^2$. The discriminant form $\langle\gamma\rangle^\perp/\langle\gamma\rangle$ has the same signature and square class as D and contains no non-trivial isotropic elements. Hence, $\langle\gamma\rangle^\perp/\langle\gamma\rangle$ is of type $2_s^{\pm1}4_t^{\pm1}$. It follows that D has order 512. The discriminant forms of order 512 and even signature containing elements of order 8 are

$$\begin{aligned}
&2_s^{\pm1}256_t^{\pm1},\ 4_s^{\pm1}128_t^{\pm1},\ 8_s^{\pm1}64_t^{\pm1},\ 2_s^{\pm3}64_t^{\pm1},\\
&16_s^{\pm1}32_t^{\pm1},\ 2_r^{\pm2}4_s^{\pm1}32_t^{\pm1},\ 2_{II}^{\pm2}4_s^{\pm1}32_t^{\pm1},\ 2_r^{\pm2}8_s^{\pm1}16_t^{\pm1},\\
&2_{II}^{\pm2}8_s^{\pm1}16_t^{\pm1},\ 2_r^{\pm1}4_s^{\pm2}16_t^{\pm1},\ 2_s^{\pm1}4_{II}^{\pm2}16_t^{\pm1},\ 2_s^{\pm5}16_t^{\pm1},\\
&2_r^{\pm1}4_s^{\pm1}8_t^{\pm2},\ 2_s^{\pm1}4_t^{\pm1}8_{II}^{\pm2},\ 4_s^{\pm3}8_t^{\pm1},\ 2_r^{\pm4}4_s^{\pm1}8_t^{\pm1},\\
&2_{II}^{\pm4}4_s^{\pm1}8_t^{\pm1}
\end{aligned}$$

with suitable s, t and signs. In the discriminant forms $2_s^{\pm1}256_t^{\pm1}$ and $4_s^{\pm1}128_t^{\pm1}$ the isotropic elements of order 8 are multiples of 2 contradicting our assumption on γ. The discriminant forms

$$\begin{aligned}
&8_s^{\pm1}64_t^{\pm1},\ 16_s^{\pm1}32_t^{\pm1},\ 2_r^{\pm2}8_s^{\pm1}16_t^{\pm1},\\
&2_{II}^{\pm2}8_s^{\pm1}16_t^{\pm1},\ 2_s^{\pm1}4_{II}^{\pm2}16_t^{\pm1},\ 2_s^{\pm5}16_t^{\pm1},\\
&4_s^{\pm3}8_t^{\pm1},\ 2_r^{\pm4}4_s^{\pm1}8_t^{\pm1},\ 2_{II}^{\pm4}4_s^{\pm1}8_t^{\pm1}
\end{aligned}$$

contain no isotropic elements of order 8 so D cannot be isomorphic to any of them. If D is of type

$$\begin{aligned}
&2_s^{\pm3}64_t^{\pm1},\ 2_r^{\pm2}4_s^{\pm1}32_t^{\pm1}, 2_{II}^{\pm2}4_s^{\pm1}32_t^{\pm1},\\
&2_r^{\pm1}4_s^{\pm2}16_t^{\pm1}, 2_r^{\pm1}4_s^{\pm1}8_t^{\pm2},
\end{aligned}$$

then any isotropic element μ of order 8 in D satisfies $4\mu \in I^\perp$ so that $\mathrm{inv}_D(e^\mu) = 0$ by Proposition 3.1.7. Finally, $2_r^{\pm1}4_s^{\pm1}8_{II}^{\pm2} \cong 2_1^{+1}4_t^{\epsilon}8_{II}^{+2}$ for some t with $t = 1 \mod 2$ and $\epsilon = \left(\frac{t}{2}\right)$.

Now suppose $\gamma \in D^2$. Again we choose a Jordan decomposition of D and write $D = A \oplus B$, where A denotes the sum over the irreducible components of exponent dividing 8 and $B \neq 0$ the sum over the remaining components. Then γ is orthogonal to B_2. Since B_2 is isotropic, we have $B_2 = \langle 4\gamma\rangle$. Hence, B is cyclic. We choose a generator β of $B \cong q_t^{\pm1}$ such that $\gamma = 2\alpha + (q/8)\beta$ for some $\alpha \in A$. An isotropic element in D is of the form $\mu + m\beta$ with $\mu \in A$ and $2 \,|\, m$. Since

$$(4\gamma, \mu + m\beta) = (q/2)m(\beta,\beta) = 0 \mod 1,$$

this implies $4\gamma \in I^\perp$. Hence, $\mathrm{inv}_D(e^\gamma) = 0$ by Proposition 3.1.7. □

The above discriminant forms, with the exception of $p^{\epsilon 2}$ and 2_{II}^{+2}, play an important role in our main result. We summarize some of their properties. First let p be an odd prime.

D	square class	signature	invariant
0	square	$0 \mod 8$	e^0
p^{-4}	square	$4 \mod 8$	$(p-1)e^0 - \sum_{\gamma \in M} e^\gamma$
$p^{\epsilon 3}$	non-square	$0 \mod 2$	$\sum_{\gamma \in M^+} e^\gamma - \sum_{\gamma \in M^-} e^\gamma$

The case $p = 2$ is more complicated.

D	square class	signature	invariant
0	square	$0 \mod 8$	e^0
2_{II}^{-4}	square	$4 \mod 8$	$e^0 - \sum_{\gamma \in M} e^\gamma$
$2_t^{+2} 4_{II}^{+2}$	square	$t = 2 \mod 4$	$\sum_{\gamma \in M^+} e^\gamma - \sum_{\gamma \in M^-} e^\gamma$
$2_1^{+1} 4_t^{\epsilon} 8_{II}^{+2}$	non-square	$1 + t = 0 \mod 2$	$\sum_{\gamma \in M^+} e^\gamma - \sum_{\gamma \in M^-} e^\gamma$

In all these cases $\mathbb{C}[D]^\Gamma$ is 1-dimensional. We wrote M for the set of isotropic elements whose order is equal to the level of D. In the indicated cases M has a canonical decomposition $M = M^+ \cup M^-$. We denote the above discriminant forms as $D_p^{x,s}$, where x is the square class and s the signature of D and the generator of the subspace of invariants as $i_p^{x,s}$.

Theorem 3.4.6. *Let D be a discriminant form of even signature s, square class x and level p^l, where p is a prime. Then the invariants of the Weil representation on $\mathbb{C}[D]$ are generated by the invariants $\uparrow_H^D (i_p^{x,s})$, where H is an isotropic subgroup of D such that $H^\perp/H$ is isomorphic to the discriminant form $D_p^{x,s}$.*

Proof. Recall that the invariants $\mathrm{inv}_D(e^\gamma)$, $\gamma \in I$ generate $\mathbb{C}[D]^\Gamma$. Let $\gamma \in I$. We will show below that at least one of the following statements applies:

i) D is a fundamental discriminant form,

ii) $\mathrm{inv}_D(e^\gamma)$ is induced from smaller discriminant forms of the same signature and square class as D, i.e. $\mathrm{inv}_D(e^\gamma)$ is a linear combination of lifts of invariants for suitable isotropic subgroups of D,

iii) $\mathrm{inv}_D(e^\gamma) = 0$.

Then the theorem follows by induction over the order of D: If $|D| = 1$, the discriminant form is fundamental. Let $|D| > 1$. If D is fundamental, there is nothing to prove. Suppose D is not fundamental. Let $\gamma \in I$. If $\mathrm{inv}_D(e^\gamma) \neq 0$, then it is a linear combination of invariants which are lifts of invariants on smaller discriminant forms. The induction hypothesis implies that the invariants on the smaller discriminant forms are induced from the fundamental invariant corresponding to D. By the transitivity of the isotropic lift (see Proposition 2.1.2) $\mathrm{inv}_D(e^\gamma)$ is a linear combination of lifts of the fundamental invariant on isotropic subgroups of D. This finishes the induction.

Now we prove that at least one of the above three statements holds. We assume that D is non-trivial. If D contains no non-trivial isotropic elements, then $\mathrm{inv}_D(e^\gamma) = 0$ for all $\gamma \in D$. We now assume that $I \neq \{0\}$. Let $\gamma \in I$. Then

$$\mathrm{inv}_D(e^0) = \rho_D(J)\,\mathrm{inv}_D(e^0) = \mathrm{inv}_D(\rho_D(J)e^0) = \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta \in I} \mathrm{inv}_D(e^\beta)$$

so that it suffices to consider $\gamma \in I \backslash \{0\}$.

We define $m = p$ if p is odd and

$$m = \begin{cases} 2 & \text{if } |D| \text{ is a square and } \mathrm{sign}(D) = 0 \mod 4, \\ 4 & \text{if } |D| \text{ is a square and } \mathrm{sign}(D) = 2 \mod 4, \\ 8 & \text{if } |D| \text{ is a non-square} \end{cases}$$

for $p = 2$.

First we consider the case that $\gamma \notin D_m$. Let n be the order of γ and $H = \langle \gamma \rangle_p$. Then for all $v = \sum_{\beta \in D} v_\beta e^\beta \in \mathbb{C}[D]^\Gamma$ we have

$$\langle v, \uparrow_H^D (\mathrm{inv}_{H^\perp/H}(e^{\gamma+H})) \rangle = \langle v, \mathrm{inv}_D(\uparrow_H^D (e^{\gamma+H})) \rangle = \sum_{\mu \in H} \langle v, \mathrm{inv}_D(e^{\gamma+\mu}) \rangle$$

$$= \sum_{\mu \in H} v_{\gamma+\mu} = \sum_{\substack{a \in \mathbb{Z}/n\mathbb{Z} \\ a = 1 \bmod n/p}} v_{a\gamma} = \sum_{\substack{a \in \mathbb{Z}/n\mathbb{Z} \\ a = 1 \bmod n/p}} \chi_D(a) v_\gamma = p v_\gamma = p \langle v, \mathrm{inv}_D(e^\gamma) \rangle$$

because $m \mid \frac{n}{p}$ so that

$$\mathrm{inv}_D(e^\gamma) = \frac{1}{p} \uparrow_H^D (\mathrm{inv}_{H^\perp/H}(e^{\gamma+H}))\,.$$

Next we consider the case $\gamma \in D_m \backslash \{0\}$. If e^γ is a linear combination of isotropic lifts for suitable isotropic subgroups, then the same holds for $\mathrm{inv}_D(e^\gamma)$ because isotropic induction and inv commute. We assume that e^γ is not a linear combination of isotropic lifts. Then $a(p, \gamma) = 1$ by Lemma 2.3.1 and Proposition 3.4.1.

Suppose χ_D is trivial. Then $m = p$ and $\mathrm{inv}_D(e^\gamma) = \mathrm{inv}_D(e^0)$ or D is of type $p^{\epsilon 2}$ with $\epsilon = \left(\frac{-1}{p}\right)$ or p^{-4} if p is odd or of type 2_{II}^{+2} or 2_{II}^{-4} if $p = 2$ (see Propositions 3.4.2 and

3.4.3). We go through the possible cases. If $\mathrm{inv}_D(e^\gamma) = \mathrm{inv}_D(e^0)$, define $H = \langle\gamma\rangle$. Then for all $v = \sum_{\beta\in D} v_\beta e^\beta \in \mathbb{C}[D]^\Gamma$ we have

$$\langle v, \uparrow_H^D (\mathrm{inv}_{H^\perp/H}(e^{0+H}))\rangle = \sum_{\beta\in H} v_\beta = v_0 + \sum_{a\in(\mathbb{Z}/p\mathbb{Z})^\times} v_{a\gamma} = v_0 + (p-1)v_\gamma$$
$$= \langle v, \mathrm{inv}_D(e^0)\rangle + (p-1)\langle v, \mathrm{inv}_D(e^\gamma)\rangle = p\langle v, \mathrm{inv}_D(e^\gamma)\rangle$$

by Proposition 3.1.3 so that $\mathrm{inv}_D(e^\gamma) = \frac{1}{p}\uparrow_H^D (\mathrm{inv}_{H^\perp/H}(e^{0+H}))$. If D is of type $p^{\epsilon 2}$ with $\epsilon = \left(\frac{-1}{p}\right)$, then $\mathbb{C}[D]^\Gamma$ is generated by the characteristic functions of the 2 maximal isotropic subgroups (see the example after Theorem 3.2.3). The same analysis holds for D of type 2_{II}^{+2}. The cases p^{-4} and 2_{II}^{-4} correspond to fundamental discriminant forms.
Finally, we assume that χ_D is non-trivial. If $m = p$ is odd, then $\mathrm{inv}_D(e^\gamma) = \mathrm{inv}_D(e^0) = 0$ or D is of type $p^{\pm 3}$ (see Propositions 3.1.4 and 3.4.2). Suppose $m = 4$. Then $\mathrm{sign}(D) = 2 \bmod 4$. If $2\gamma = 0$, then $v_\gamma = \chi_D(3)v_{3\gamma} = -v_\gamma = 0$ for all $v = \sum_{\beta\in D} v_\beta e^\beta \in \mathbb{C}[D]^\Gamma$ which implies $\mathrm{inv}_D(e^\gamma) = 0$. If γ has order 4, then $\mathrm{inv}_D(e^\gamma) = 0$ or D is of type $2_t^{+2}4_{II}^{+2}$ (see Proposition 3.4.4). The case $m = 8$ is analogous and uses Proposition 3.4.5. □

A few comments are in order. It is possible that more than one of the conditions i), ii) and iii) applies (see e.g. Proposition 3.3.6). A consequence of the theorem is that the invariants are defined over $\mathbb{Z}$. A more direct proof of this fact is given in [24]. The theorem extends Theorem 4.11 in [50] to $p = 2$.

We describe some examples. Let p be an odd prime and D a discriminant form of even signature. If $|D| = p$, then D is not fundamental and $\dim(\mathbb{C}[D]^\Gamma) = 0$. Suppose $|D| = p^2$. Then D is not fundamental and there are three possibilities. If D has level p and is anisotropic, then $\dim(\mathbb{C}[D]^\Gamma) = 0$. If D has level p and is isotropic, then D has two non-trivial isotropic subgroups H_i of order p with fundamental quotients $H_i^\perp/H_i \cong 0$. They generate $\mathbb{C}[D]^\Gamma$ which has dimension 2. If D has level p^2, then D has a unique non-trivial isotropic subgroup H with fundamental quotient $H^\perp/H \cong 0$. It follows $\dim(\mathbb{C}[D]^\Gamma) = 1$. Finally, let $|D| = p^3$. We only consider the case that D has level p. Then D is fundamental. Nevertheless D has non-trivial isotropic subgroups H_i of order p. Here the quotients $H_i^\perp/H_i$ have order p so that no non-trivial invariants can be induced from them.

Corollary 3.4.7. *Let D be a discriminant form of even signature s, square class x and level p^l, where p is a prime. Suppose $|D| < |D_p^{x,s}|$. Then* $\dim \mathbb{C}[D]^\Gamma = 0$.

Proof. If there were non-trivial invariants in $\mathbb{C}[D]$, they would be induced from $D_p^{x,s}$. But this is impossible. □

3.5 Applications

The above results have several applications. For example, the dimension of the space of weight-2 cusp forms transforming under the Weil representation has contributions coming from the invariants. Furthermore, the theta expansion gives an isomorphism between modular forms for the Weil representation and Jacobi forms of lattice index. The invariants of the Weil representation can be used to give simple generating sets for Jacobi forms of singular weight. Another example comes from orthogonal modular forms. Borcherds' additive theta lift (Theorem 14.3 in [7]) maps the invariants of the Weil representation to orthogonal modular forms of singular weight. This allows to study orthogonal modular forms of singular weight with a special boundary behaviour. We will describe the first two examples in more detail.

A dimension formula for cusp forms of weight 2

Let D be a discriminant form of level p, where p is a prime. We give an explicit formula for the dimension of the space $\mathrm{S}_2(D)$ of cusp forms of weight 2 for the Weil representation ρ_D.

Let ρ be a finite-dimensional representation of $\mathrm{SL}_2(\mathbb{Z})$ with finite image. Then the dimension of the space of modular forms for ρ of weight at least 2 can be determined by means of the Selberg trace formula or the Riemann-Roch theorem (see e.g. [65], [9] and [31]). In weight 2 there is a contribution coming from the invariants of ρ. We will follow Freitag's approach [31].

Let D be a discriminant form of prime level. We assume that D is of type $p^{\epsilon n}$ with n even. The argument for odd n is similar. Then $\mathrm{sign}(D) = 0 \bmod 4$ so that Z acts as $\rho_D(Z)e^\gamma = e^{-\gamma}$. The space $V \subset \mathbb{C}[D]$ spanned by the elements $e^\gamma + e^{-\gamma}$, $\gamma \in D$ is invariant under ρ_D. Let ρ be the restriction of ρ_D to V and $d = \dim(V)$. For a complex $d \times d$-matrix M of finite order with eigenvalues $e(x_i)$, $0 \leq x_i < 1$ define

$$\alpha(M) = \sum_{i=1}^{d} x_i \,,$$

in particular

$$\alpha(M) = \begin{cases} \dfrac{d}{4} - \dfrac{\mathrm{tr}(M)}{4} & \text{if } M^2 = I, \\[2ex] \dfrac{d}{3} - \dfrac{1}{3}\,\mathrm{Re}(\mathrm{tr}(M^{-1})) + \dfrac{1}{3\sqrt{3}}\,\mathrm{Im}(\mathrm{tr}(M^{-1})) & \text{if } M^3 = I. \end{cases}$$

Then the dimension of $\mathrm{S}_2(D)$ is given by

$$\dim \mathrm{S}_2(D) = \frac{d}{6} + d - \alpha\big(e(1/2)\rho(-J)\big) - \alpha\big((e(1/3)\rho(-JT))^{-1}\big) - \alpha\big(\rho(T)\big) \\ - |\{\gamma \in D/\{\pm 1\} \,|\, \mathrm{q}(\gamma) = 0 \mod 1\}| + \dim \mathbb{C}[D]^{\Gamma}$$

(see Theorem 6.1 in [31]). We can evaluate this expression using Theorem 3.2.3.

Theorem 3.5.1. *Let D be a discriminant form of prime level p and type $p^{\epsilon n}$ with n even. Then* $\dim \mathrm{S}_2(D) = 0$ *if* $p \leq 3$ *and*

$$\dim \mathrm{S}_2(D) = \frac{p^n+5}{24} - \frac{p^{n-1}}{4} - \epsilon \left(\frac{-1}{p}\right)^{n/2} \frac{p-5}{4} p^{(n-2)/2} + \frac{p^{n-1}-p}{p^2-1}$$

if $p > 3$.

Proof. Since $\Gamma(p)$ acts trivial in the Weil representation ρ_D, the components of an element in $S_2(D)$ are cusp forms for $\Gamma(p)$. The spaces $S_2(\Gamma(p))$ are trivial for $p \leq 3$ so that $\dim S_2(D) = 0$ in these cases. Suppose $p > 3$. Clearly

$$d = \frac{p^n - 1}{2} + 1 = \frac{p^n+1}{2}\,.$$

Proposition 1.1.1 implies

$$|\{\gamma \in D/\{\pm 1\} \,|\, \mathrm{q}(\gamma) = 0 \mod 1\}| = \frac{N(p^{\epsilon n}, 0) - 1}{2} + 1 \\ = \frac{p^{n-1}+1}{2} + \epsilon \left(\frac{-1}{p}\right)^{n/2} \frac{p-1}{2} p^{(n-2)/2}$$

and

$$\alpha(\rho(T)) = \sum_{j=0}^{p-1} \frac{j}{p} \, |\{\gamma \in D/\{\pm 1\} \mid \mathrm{q}(\gamma) = j/p \mod 1\}| \\ = \frac{1}{2} \sum_{j=1}^{p-1} \frac{j}{p} \, N(p^{\epsilon n}, j/p) \\ = \frac{p-1}{4} \big(p^{n-1} - \epsilon \left(\frac{-1}{p}\right)^{n/2} p^{(n-2)/2} \big)\,.$$

Since $e(1/2)\rho(-J)$ has order 2, we can apply the above formula in order to calculate

$\alpha(e(1/2)\rho(-J))$. For the trace of $e(1/2)\rho(-J)$ we find

$$\begin{aligned}
\operatorname{tr}(e(1/2)\rho(-J)) &= -\frac{1}{4}\sum_{\gamma\in D}(\rho(-J)(e^\gamma + e^{-\gamma}), e^\gamma + e^{-\gamma}) \\
&= -\frac{e(-\operatorname{sign}(D)/8))}{4p^{n/2}} \sum_{\beta,\gamma\in D} e(-(\beta,\gamma))(e^\beta + e^{-\beta}, e^\gamma + e^{-\gamma}) \\
&= -\frac{e(-\operatorname{sign}(D)/8))}{2p^{n/2}} \sum_{\gamma\in D}\{e(2\,\mathrm{q}(\gamma)) + e(-2\,\mathrm{q}(\gamma))\} \\
&= -\frac{e(-\operatorname{sign}(D)/8))}{p^{n/2}}\, e(\operatorname{sign}(D)/8))p^{n/2} \\
&= -1,
\end{aligned}$$

where we used Milgram's formula to evaluate the last sum. Hence,

$$\alpha\big(e(1/2)\rho(-J)\big) = \frac{d}{4} - \frac{\operatorname{tr}(e(1/2)\rho(-J))}{4} = \frac{p^n+3}{8}\,.$$

Similarly, we find

$$\alpha\big((e(1/3)\rho(-JT))^{-1}\big) = \frac{p^n+3}{6}\,.$$

Finally, $\dim \mathbb{C}[D]^\Gamma$ is given in Theorem 3.2.3. Putting all the contributions together we obtain the desired formula for the dimension of $\mathrm{S}_2(D)$. □

Jacobi forms of singular weight

The space of Jacobi forms $J_{k,L}$ of lattice index L and singular weight $k = \operatorname{rk}(L)/2$, is naturally isomorphic to the space of invariants $\mathbb{C}[L'/L]^{\mathrm{Mp}_2(\mathbb{Z})}$. This allows us to write down a generating set for this space.

Jacobi forms of lattice index are natural generalizations of Jacobi forms in one variable [28]. They were introduced by Gritsenko [35]. Classical examples are Jacobi theta functions. We recall the definition of Jacobi forms of lattice index and describe some of their properties (cf. e.g. [66], [36]).

Let L be a positive-definite even lattice of rank n. Then $\mathrm{Mp}_2(\mathbb{Z})$ acts from the right on the pairs $(\lambda,\mu) \in L\times L$. The corresponding semidirect product $J_L = \mathrm{Mp}_2(\mathbb{Z}) \ltimes (L\times L)$ is the Jacobi group of lattice index L. Recall that the product of two elements in J_L is given by

$$\begin{aligned}
&\big((M_1,\omega_1(\tau)),(\lambda_1,\mu_1)\big)\,\big((M_2,\omega_2(\tau)),(\lambda_2,\mu_2)\big)\\
&\qquad = \big((M_1M_2,\omega_1(M_2\tau)\omega_2(\tau)),(\lambda_1,\mu_1)M_2 + (\lambda_2,\mu_2)\big)\,.
\end{aligned}$$

We identify $\mathrm{Mp}_2(\mathbb{Z})$ with the subgroup $\mathrm{Mp}_2(\mathbb{Z}) \ltimes (0 \times 0)$ and $L \times L$ with $1 \ltimes (L \times L)$.

Now let $k \in \frac{1}{2}\mathbb{Z}$. We define an action of the Jacobi group J_L on the functions $\phi : \mathbb{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C}) \to \mathbb{C}$ by

$$\phi|_k[(M,\omega)](\tau,z) = \phi\left(M\tau, \frac{z}{c\tau+d}\right)\omega(\tau)^{-2k} e\left(\frac{-cz^2/2}{c\tau+d}\right)$$
$$\phi|_k[(\lambda,\mu)](\tau,z) = \phi\left(\tau, z+\lambda\tau+\mu\right) e\left(\tau\lambda^2/2 + (\lambda,z)\right),$$

where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, $\lambda, \mu \in L$. A *Jacobi form* of weight k and index L is a holomorphic function $\phi : \mathbb{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C}) \to \mathbb{C}$ which is invariant under the action of J_L and possesses a Fourier expansion of the form

$$\phi(\tau,z) = \sum_{\substack{m\in\mathbb{Z},\, \alpha\in L' \\ m\geq(\alpha,\alpha)/2}} c(m,\alpha) e(m\tau + (\alpha,z)) .$$

We denote the space of Jacobi forms of weight k and lattice index L by $J_{k,L}$. A Jacobi form $\phi \in J_{k,L}$ has a unique theta expansion

$$\phi(z,\tau) = \sum_{\gamma\in L'/L} \vartheta_\gamma(z,\tau) f_\gamma(\tau),$$

where

$$\vartheta_\gamma(\tau,z) = \sum_{\alpha\in\gamma+L} e(\tau(\alpha,\alpha)/2 + (\alpha,z))$$

is the Jacobi theta function of the coset $\gamma + L$ and $f(\tau) = \sum_{\gamma\in L'/L} f_\gamma(\tau) e^\gamma$ is a modular form for the dual Weil representation $\overline{\rho}_{L'/L}$ of L'/L. We obtain a map

$$J_{k,L} \to \mathrm{M}_{k-n/2}(L'/L),$$

which is actually an isomorphism. This implies that $J_{k,L}$ is trivial for $k < n/2$. The weight $k = n/2$ is called *singular* weight. In this case we have an isomorphism

$$\begin{aligned} \mathbb{C}[L'/L]^{\mathrm{Mp}_2(\mathbb{Z})} &\xrightarrow{\varphi_L} J_{n/2,L} \\ \sum_{\gamma\in L'/L} v_\gamma e^\gamma &\longmapsto \sum_{\gamma\in L'/L} v_\gamma \vartheta_\gamma \end{aligned}$$

(cf. also Theorem 5 in [66]). Hence, $J_{n/2,L}$ is trivial for odd n. For even n we can generate $J_{n/2,L}$ by relatively few functions. Note that the space of invariants for $\overline{\rho}_{L'/L}$ is identical to the space of invariants for $\rho_{L'/L}$ since by Theorem 3.4.6 it has a basis consisting of elements with coefficients in $\mathbb{Z}$. Applying Theorem 3.4.6 we obtain:

Theorem 3.5.2. *Let L be a positive-definite even lattice of rank n and level N. Suppose n is even. For $p\,|\,N$ we denote the square class and the signature of the p-adic component of L'/L by x_p resp. s_p. Let $\mathcal{L}$ be the set of all overlattices $M \supset L$ such that the p-adic component of M'/M is isomorphic to $D_p^{x_p,s_p}$ for all $p\,|\,N$. Then*

$$J_{n/2,L} = \sum_{M\in\mathcal{L}} \mathbb{C}\Bigg(\sum_{\gamma\in M'/M} v_\gamma \vartheta_{M,\gamma} \Bigg),$$

where $\sum_{\gamma\in M'/M} v_\gamma e^\gamma \in \mathbb{C}[M'/M]^{\mathrm{SL}_2(\mathbb{Z})}$ is the invariant corresponding to the product $\prod_{p\,|\,N} i_p^{x_p,s_p}$.

Proof. For $M \in \mathcal{L}$ we have $L \subset M \subset M' \subset L'$ and M/L is an isotropic subgroup of L'/L. Let $v = \sum_{\gamma\in M'/M} v_\gamma e^\gamma \in \mathbb{C}[M'/M]^{\mathrm{SL}_2(\mathbb{Z})}$. Then

$$\uparrow_{M/L}^{L'/L}(v) = \sum_{\gamma\in M'/M} v_\gamma \uparrow_{M/L}^{L'/L}(e^\gamma) = \sum_{\eta+M\in M'/M} v_{\eta+M} \sum_{\beta\in M/L} e^{\eta+\beta}$$

so that

$$\begin{aligned}
\varphi_L(\uparrow_{M/L}^{L'/L}(v)) &= \sum_{\eta+M\in M'/M} v_{\eta+M} \sum_{\beta\in M/L} \vartheta_{L,\eta+\beta} \\
&= \sum_{\eta+M\in M'/M} v_{\eta+M} \sum_{\beta\in M/L} \sum_{\alpha\in\eta+\beta+L} e(\tau(\alpha,\alpha)/2 + (\alpha,z)) \\
&= \sum_{\eta+M\in M'/M} v_{\eta+M} \sum_{\alpha\in\eta+M} e(\tau(\alpha,\alpha)/2 + (\alpha,z)) \\
&= \sum_{\gamma\in M'/M} v_\gamma \vartheta_{M,\gamma} \\
&= \varphi_M(v)
\end{aligned}$$

because

$$\eta + \bigcup_{\beta\in M/L} (\beta + L) = \eta + M\,.$$

Hence, the diagram

$$\begin{array}{ccc}
\mathbb{C}[L'/L]^{\mathrm{SL}_2(\mathbb{Z})} & \xrightarrow{\varphi_L} & J_{n/2,L} \\
\Big\uparrow{\scriptstyle \uparrow_{M/L}^{L'/L}} & & \Big\uparrow \\
\mathbb{C}[M'/M]^{\mathrm{SL}_2(\mathbb{Z})} & \xrightarrow{\varphi_M} & J_{n/2,M}
\end{array}$$

commutes. The assertion now follows from Theorem 3.4.6. $\square$

Chapter 4

The basis problem for the Weil representation

In this chapter we will show that for a discriminant form D of even signature $\mathrm{sign}(D) = m \bmod 8$ ($m \in \mathbb{Z}_{>0}$) and an integer $k \geq m/2$ the space of cusp forms $\mathrm{S}_k(D)$ is generated by the theta series in the genus $II_{m,0}(D)$ when m is sufficiently large compared to the p-ranks of D.

This chapter is based on the preprint [51].

4.1 Vector-valued Hecke operators

First, we want to define vector-valued Hecke operators. They were introduced by Bruinier and Stein in [14] and naturally appear when we later use the doubling method. We will study some of their properties and describe their kernel functions.

Let D be a discriminant form of even signature and level N. We define Hecke operators $T(l^2)$ acting on $\mathrm{M}_k(D)$ (see [14]). In order to do so we extend the right action given by the inverse of the Weil representation to certain matrices in $\mathrm{Mat}_2(\mathbb{Z})$. Let l be a non-negative integer and $\alpha = \left(\begin{smallmatrix} l^2 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Mat}_2(\mathbb{Z})$. We define

$$\rho_D(\alpha)^{-1} e^{\gamma} = e^{l\gamma}.$$

For any $\delta = A\alpha B \in \Gamma\alpha\Gamma$ we put

$$\rho_D(\delta)^{-1} e^{\gamma} = \rho_D(B^{-1})\rho_D(\alpha)^{-1}\rho_D(A^{-1}) e^{\gamma}.$$

It is shown in [14] that $\rho_D(\delta)^{-1}$ is well-defined and that

$$\rho_D(A\delta B)^{-1} e^{\gamma} = \rho_D(B)^{-1}\rho_D(\delta)^{-1}\rho_D(A)^{-1} e^{\gamma} = \rho_D(B^{-1})\rho_D(\delta)^{-1}\rho_D(A^{-1}) e^{\gamma}$$

for all $\delta \in \Gamma\alpha\Gamma$ and $A, B \in \Gamma$. Furthermore, $\beta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & l^2 \end{smallmatrix}\right) \in \Gamma\alpha\Gamma$, in fact $\beta = -J\alpha J$ and

$$\rho_D(\beta)^{-1}e^\gamma = \sum_{\substack{\mu \in D \\ l\mu = \gamma}} e^\mu.$$

The element β satisfies

$$\langle \rho_D(\alpha)^{-1}v, w\rangle = \langle v, \rho_D(\beta)^{-1}w\rangle \quad \text{and}$$
$$\langle \rho_D(\beta)^{-1}v, w\rangle = \langle v, \rho_D(\alpha)^{-1}w\rangle.$$

When $(l, N) = 1$, we find that for $\delta \in \Gamma\alpha\Gamma$

$$\rho_D(\delta)^{-1} = \chi_D(l)\rho_D(\tilde{\delta})^{-1} = \chi_D(l)\rho_D(\tilde{\delta}^{-1}), \tag{4.1.1}$$

where $\tilde{\delta} \in \Gamma$ is any representative of $l^{-1}\delta \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. The following lemma is well-known (see e.g. [30, Hilfssatz IV.1.12])

Lemma 4.1.1. *For $l, m \in \mathbb{Z}$ with $(l, m) \neq (0, 0)$ we have the equality*

$$\Gamma \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} \Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}) \mid ad - bc = lm, \gcd(a, b, c, d) = \gcd(l, m) \right\}.$$

We will sometimes, when convenient, simply write $(\cdot, \cdot)$ for $\gcd(\cdot, \cdot)$. Denote by

$$M_l := \Gamma\alpha\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}) \mid ad - bc = l^2, \gcd(a, b, c, d) = 1 \right\}.$$

Now let

$$M_l = \Gamma\alpha\Gamma = \bigcup_i \Gamma \cdot \delta_i$$

be a disjoint right coset decomposition. We define the Hecke operator $T(l^2)$ on modular forms $f \in \mathrm{M}_k(D)$ by

$$T(l^2)f := l^{k-2} \sum_i \rho_D(\delta_i)^{-1} f|_k[\delta_i].$$

Then

Theorem 4.1.2 (Theorem 5.6, [14]). *For any positive integer l, the Hecke operator $T(l^2)$ is a linear operator on $\mathrm{M}_k(D)$ taking cusp forms to cusp forms. It is self-adjoint with respect to the Petersson scalar product. Moreover, if l, m are coprime, then*

$$T(l^2)T(m^2) = T(l^2m^2).$$

The operators $T(l^2)$ with l coprime to N behave analogously to the classical Hecke operators for $\mathrm{SL}_2(\mathbb{Z})$.

Proposition 4.1.3. *The algebra generated by all Hecke operators $T(l^2)$ with $(l, N) = 1$ is a commutative algebra of self-adjoint operators. Hence, there exists a basis of $\mathrm{S}_k(D)$ consisting of simultaneous eigenforms for it. Let f be a simultaneous eigenform with eigenvalues $\lambda(l^2)$. The L-series*

$$L(f, s) := \sum_{\substack{l=1 \\ (l,N)=1}}^{\infty} \frac{\lambda(l^2)}{l^s}$$

converges for $\mathrm{Re}(s) > k$ and has an Euler-product

$$L(f, s) = \prod_{p \nmid N} \frac{(1 - \chi_D(p)p^{k-2-s})(1 + \chi_D(p)p^{k-1-s})}{1 - (\lambda(p^2) + \chi_D(p)(1-p)p^{k-2})p^{-s} + p^{2k-2-2s}}.$$

Proof. Let $(l, N) = 1$ and $\widetilde{M_l} := \{M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Mat}_2(\mathbb{Z}) \mid \det(M) = l^2\}$. We define operators $\widetilde{T}(l^2)$ by

$$\widetilde{T}(l^2)f := l^{k-2} \sum_{\delta \in \Gamma \backslash \widetilde{M_l}} \rho_D(\delta)^{-1} f|_k[\delta],$$

where $\rho_D(\delta)^{-1}$ acts as $\chi_D(l)\rho_D(\tilde{\delta}^{-1})$ and $\tilde{\delta} \in \Gamma$ is any representative of $l^{-1}\delta \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. By equation (4.1.1) this extends the previously defined action of M_l. Clearly

$$\widetilde{M_l} = \bigcup_{d|l} \frac{l}{d} M_d$$

so that

$$\widetilde{T}(l^2) = \sum_{d|l} \chi_D\left(\frac{l}{d}\right)\left(\frac{l}{d}\right)^{k-2} T(d^2),$$

which implies $T(p^{2r}) = \widetilde{T}(p^{2r}) - \chi_D(p)p^{k-2}\widetilde{T}(p^{2(r-1)})$ for a prime p. Therefore, the operators $T(l^2)$ and $\widetilde{T}(l^2)$ generate the same algebra of operators. Furthermore, as in the classical scalar case, we can show that for $r \geq 2$ we have

$$\widetilde{T}(p^{2r}) = \widetilde{T}(p^{2(r-1)})(\widetilde{T}(p^2) - \chi_D(p)p^{k-1}) - p^{2k-2}\widetilde{T}(p^{2(r-2)}) \tag{4.1.2}$$

and so this algebra is a commutative algebra of self-adjoint operators. (More details of this Hecke algebra can be found in [48].) This implies that $\mathrm{S}_k(D)$ has a basis consisting of simultaneous eigenforms. Let f be an eigenform with eigenvalues $\lambda(l^2)$ for $T(l^2)$ and $\widetilde{\lambda}(l^2)$ for $\widetilde{T}(l^2)$. We define

$$\widetilde{L}(f, s) := \sum_{\substack{l=1 \\ (l,N)=1}}^{\infty} \frac{\widetilde{\lambda}(l^2)}{l^s}.$$

Then we have

$$\begin{aligned}
L(f,s) &= \prod_{p\nmid N}\sum_{r=0}^{\infty}\frac{\lambda(p^{2r})}{p^{rs}}\\
&= \prod_{p\nmid N}\left(\sum_{r=0}^{\infty}\frac{\widetilde{\lambda}(p^{2r})}{p^{rs}} - \chi_D(p)p^{k-2-s}\sum_{r=0}^{\infty}\frac{\widetilde{\lambda}(p^{2r})}{p^{rs}}\right)\\
&= \prod_{p\nmid N}(1-\chi_D(p)p^{k-2-s})\left(\sum_{r=0}^{\infty}\frac{\widetilde{\lambda}(p^{2r})}{p^{rs}}\right)\\
&= \widetilde{L}(f,s)\prod_{p\nmid N}(1-\chi_D(p)p^{k-2-s}).
\end{aligned}$$

The convergence of the latter product is clear for $\mathrm{Re}(s) > k$, as it the reciprocal of a Dirichlet series. We want to deduce the convergence of $\widetilde{L}(f,s)$ from the scalar situation. Let $\gamma \in D$ be of order n and $\chi : (\mathbb{Z}/n\mathbb{Z})^\times \to \mathbb{C}^\times$ a character. Define

$$v_{\gamma,\chi} \coloneqq \sum_{x\in(\mathbb{Z}/n\mathbb{Z})^\times}\chi(x)^{-1}e^{x\gamma}.$$

Let $R_a \in \mathrm{SL}_2(\mathbb{Z})$ be any preimage of $\left(\begin{smallmatrix} a^{-1} & 0\\ 0 & a\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Then

$$\left\{R_a\left(\begin{smallmatrix} a & bN\\ 0 & d\end{smallmatrix}\right) \mid ad = l^2, 0 \le b < d\right\}$$

is a system of representatives for $\Gamma\backslash\widetilde{M}_l$, but also for

$$\Gamma(N)\backslash\{M\in\mathrm{Mat}_2(\mathbb{Z}) \mid \det(M) = l^2, M = \left(\begin{smallmatrix} 1 & 0\\ 0 & l^2\end{smallmatrix}\right) \bmod N\}.$$

Using this system of representatives we compute

$$\langle\widetilde{T}(l^2)f, v_{\gamma,\chi}\rangle = \chi(l)^{-1}T^{\Gamma(N)}(l^2)\langle f, v_{\gamma,\chi}\rangle,$$

where $T^{\Gamma(N)}(l^2)$ is the standard Hecke operator on $\mathrm{S}_k(\Gamma(N))$ (cf. [71, Theorem 31 (ii)]). Since the elements of the form $v_{\gamma,\chi}$ generate $\mathbb{C}[D]$, we find a pair (γ,χ) such that

$$g \coloneqq \langle f, v_{\gamma,\chi}\rangle \neq 0$$

and $g \in \mathrm{S}_k(\Gamma(N))$ is a simultaneous eigenform with eigenvalues $\chi(l)\widetilde{\lambda}(l^2)$. Now the convergence of $\widetilde{L}(f,s)$ follows from the scalar case (see for example [57]). Again we write

$$\sum_{\substack{l=1\\(l,N)=1}}^{\infty}\frac{\widetilde{\lambda}(l^2)}{l^s} = \prod_{p\nmid N}\sum_{r=0}^{\infty}\frac{\widetilde{\lambda}(p^{2r})}{p^{rs}}.$$

From (4.1.2) we obtain

$$\begin{aligned}
&\sum_{r=0}^{\infty}\frac{\widetilde{\lambda}(p^{2r})}{p^{rs}}\\
&\quad= 1+\frac{\widetilde{\lambda}(p^2)}{p^s}+(\widetilde{\lambda}(p^2)-\chi_D(p)p^{k-1})\cdot\sum_{r=2}^{\infty}\frac{\widetilde{\lambda}(p^{2(r-1)})}{p^{rs}}-p^{2k-2}\cdot\sum_{r=2}^{\infty}\frac{\widetilde{\lambda}(p^{2(r-2)})}{p^{rs}}\\
&\quad= 1+\chi_D(p)p^{k-1-s}+\frac{\widetilde{\lambda}(p^2)-\chi_D(p)p^{k-1}}{p^s}\cdot\sum_{r=0}^{\infty}\frac{\widetilde{\lambda}(p^{2r})}{p^{rs}}-p^{2k-2-2s}\cdot\sum_{r=0}^{\infty}\frac{\widetilde{\lambda}(p^{2r})}{p^{rs}},
\end{aligned}$$

which implies

$$\sum_{r=0}^{\infty}\frac{\widetilde{\lambda}(p^{2r})}{p^{rs}}=\frac{1+\chi_D(p)p^{k-1-s}}{1-(\widetilde{\lambda}(p^2)-\chi_D(p)p^{k-1})p^{-s}+p^{2k-2-2s}}.$$

The theorem now follows from $\widetilde{\lambda}(p^2)=\lambda(p^2)+\chi_D(p)p^{k-2}$. □

We also want to study the behaviour of the Hecke operator $T(p^{2r})$ for $p \mid N$ and $r\in\mathbb{Z}_{\geq 0}$. Let $x\in\mathbb{Q}/\mathbb{Z}$ with $Nx = 0 \bmod 1$. Then

$$\chi_x\left(\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\right)=e(bx)$$

is a character on $\Gamma_1(N)$. For $p \mid N$ and $r\in\mathbb{Z}_{\geq 0}$ we define an operator $T^x(p^{2r}):\mathrm{M}_k(\Gamma_1(N),\chi_{p^{2r}x})\to\mathrm{M}_k(\Gamma_1(N),\chi_x)$ by

$$T^x(p^{2r})f:=p^{rk-2r}\sum_{b=0}^{p^{2r}-1}e(-bx)f|_k\left[\left(\begin{smallmatrix} 1 & b\\ 0 & p^{2r}\end{smallmatrix}\right)\right].$$

It is not difficult to verify that this is well-defined.

Proposition 4.1.4. *Let D be a discriminant form of even signature, p a prime and $r\geq 1$. Let $f\in\mathrm{M}_k(D)$ and $\gamma\in D$ with $\gamma\notin D^p$ and if $p=2$, also $\gamma\notin D^{2*}$. Then*

$$\langle T(p^{2r})f,e^\gamma\rangle=T^{\mathrm{q}(\gamma)}(p^{2r})\langle f,e^{p^r\gamma}\rangle.$$

Proof. A system of representatives δ_i of the right coset decomposition of $M_{p^{2r}}$ is given by

$$\left\{\delta_{s,b}=\begin{pmatrix} p^s & b\\ 0 & p^{2r-s}\end{pmatrix}\mid 0\leq s\leq 2r, 0\leq b<p^{2r-s}\text{ and }(b,p)=1\text{ if }0<s<2r\right\}$$

so that

$$\begin{aligned}
\langle T(p^{2r})f,e^\gamma\rangle &= p^{rk-2r}\langle\rho_D(\alpha)^{-1}f|_k[\alpha],e^\gamma\rangle+p^{rk-2r}\sum_{s=1}^{2r-1}\sum_{\substack{b=0\\(b,p)=1}}^{p^{2r-s}-1}\langle\rho_D(\delta_{s,b})^{-1}f|_k[\delta_{s,b}],e^\gamma\rangle\\
&\quad+p^{rk-2r}\sum_{b=0}^{p^{2r}-1}\langle\rho_D(\delta_{0,b})^{-1}f|_k[\delta_{0,b}],e^\gamma\rangle.
\end{aligned}$$

Recall that

$$\rho_D(\beta)^{-1}e^\gamma = \sum_{\substack{\mu\in D\\ p^r\mu=\gamma}} e^\mu$$

and so

$$\langle \rho_D(\alpha)^{-1} f|_k[\alpha], e^\gamma\rangle = \langle f|_k[\alpha], \rho_D(\beta)^{-1}e^\gamma\rangle = 0$$

because $\gamma \notin D^p$. For a given $s < 2r$ and b, there exist $x, y \in \mathbb{Z}$ such that $xb - yp^s = 1$. Hence, we can write

$$\begin{pmatrix} p^s & b \\ 0 & p^{2r-s}\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & p^{2r-s}x\end{pmatrix}\begin{pmatrix} p^{2r} & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} x & y \\ p^s & b\end{pmatrix}.$$

Let $0 < s < 2r$. Similar to $s = 2r$, using the explicit formula (1.2.1) for the action of an arbitrary element in the Weil representation we obtain

$$\begin{aligned}\langle \rho_D(\delta_{s,b})^{-1} f|_k[\delta_{s,b}], e^\gamma\rangle &= \langle f|_k[\delta_{s,b}], \rho_D(\left(\begin{smallmatrix} 0 & 1 \\ -1 & p^{2r-s}x\end{smallmatrix}\right))\rho_D(\beta)^{-1}\rho_D(\left(\begin{smallmatrix} x & y \\ p^s & b\end{smallmatrix}\right))e^\gamma\rangle \\ &= \langle f|_k[\delta_{s,b}], \rho_D(\left(\begin{smallmatrix} 0 & 1 \\ -1 & p^{2r-s}x\end{smallmatrix}\right))\rho_D(\beta)^{-1} \\ &\qquad \xi\frac{\sqrt{|D_{p^s}|}}{\sqrt{|D|}}\sum_{\mu\in D^{p^s*}} e(x\,\mathrm{q}_{p^s}(\mu))e(y(\mu,\gamma))e(by\,\mathrm{q}(\gamma))e^{b\gamma+\mu}\rangle \\ &= 0\end{aligned}$$

because of the following reasoning: Suppose that $p^r\gamma' = b\gamma + \mu$ for some $\gamma' \in D$. If p is odd, then $D^{p^s*} = D^{p^s}$ and $\mu = p^s\mu'$. But then also

$$b\gamma = p(p^{r-1}\gamma' - p^{s-1}\mu').$$

Since $(b, p) = 1$, this contradicts $\gamma \notin D^p$. If $p = 2$, first consider $s = 1$. Then $b\gamma = 2^r\gamma' - \mu \in D^{2*}$ because D^{2*} is a coset of D^2. It is not difficult to see that then also $\gamma \in D^{2*}$ because b is odd. If $s > 1$, recall that $D^{2^s*} \subset D^{2^{s-1}}$. Hence, $\mu = 2^{s-1}\mu'$ and $b\gamma = 2(2^{r-1}\gamma' - 2^{s-2}\mu') \in D^2$ and since b is odd, also $\gamma \in D^2$.
Finally, $\delta_{0,b} = \beta\left(\begin{smallmatrix} 1 & b \\ 0 & 1\end{smallmatrix}\right)$ implies

$$\begin{aligned}\langle \rho_D(\delta_{0,b})^{-1} f|_{\delta_{0,b}}, e^\gamma\rangle &= \langle f|_{\delta_{0,b}}, \rho_D(\alpha)^{-1}\rho_D(\left(\begin{smallmatrix} 1 & b \\ 0 & 1\end{smallmatrix}\right))e^\gamma\rangle \\ &= e(-b\,\mathrm{q}(\gamma))\langle f|_{\delta_{0,b}}, e^{p^r\gamma}\rangle.\end{aligned}$$

Therefore, we find

$$\langle T(p^{2r})f, e^\gamma\rangle = p^{rk-2r}\sum_{b=0}^{p^{2r}-1} e(-b\,\mathrm{q}(\gamma))\langle f, e^{p^r\gamma}\rangle|_k[\delta_{0,b}].$$

□

Let $\gamma, \mu \in D$ and p be a prime. We call the projection of γ to the p-adic component of D the p-adic component of γ. For a finite set of primes P denote by $\gamma_P^\mu \in D$ the element whose p-adic components are equal to those of μ for all $p \in P$ and equal to those of γ for all other p. For example $\gamma_\emptyset^\mu = \gamma$ and $\gamma_P^\mu = \mu$ if P contains all primes $p \mid N$. For $a \in \mathbb{Z}$ we have $a\gamma_P^\mu = (a\gamma)_P^{a\mu}$. We define

$$v_{\gamma,\mu,P} := \sum_{S \subset P} (-1)^{|S|} e^{\gamma_S^\mu}.$$

Then we have

Corollary 4.1.5. *Let D be a discriminant form of even signature, P a finite set of prime numbers and $\gamma, \mu \in D$. Assume that $\left(\prod_{p\in P} p\right)(\gamma - \mu) = 0$, $\mathrm{q}(\gamma) = \mathrm{q}(\mu) \bmod 1$ and for all $p \in P$ assume $\gamma, \mu \notin D^p$ and if $2 \in P$, also $\gamma, \mu \notin D^{2*}$. Let $s \in \mathbb{C}$ and $f \in \mathrm{S}_k(D)$ and assume that*

$$\prod_{p\in P} \left(\sum_{r=0}^{\infty} \frac{T(p^{2r})}{p^{rs}} \right) f$$

converges. Then

$$\langle \prod_{p\in P} \left(\sum_{r=0}^{\infty} \frac{T(p^{2r})}{p^{rs}} \right) f, v_{\gamma,\mu,P} \rangle = \langle f, v_{\gamma,\mu,P} \rangle.$$

Proof. We prove the statement by induction on $|P|$: Note that for $P = \emptyset$ there is nothing to prove.
So let $|P| \geq 1$ and assume that the assertion holds for all sets smaller than P. Let $p \in P$. We have

$$v_{\gamma,\mu,P} = v_{\gamma,\mu^\gamma_{\{p\}},P\setminus\{p\}} - v_{\gamma^\mu_{\{p\}},\mu,P\setminus\{p\}}.$$

The pairs $(\gamma, \mu^\gamma_{\{p\}})$ and $(\gamma^\mu_{\{p\}}, \mu)$ satisfy the conditions of the corollary for the set of primes $P \setminus \{p\}$ and so by the induction hypothesis we have

$$\langle \prod_{q\in P\setminus\{p\}} \left(\sum_{r=0}^{\infty} \frac{T(q^{2r})}{q^{rs}} \right) f, v_{\gamma,\mu,P} \rangle = \langle f, v_{\gamma,\mu,P} \rangle.$$

In fact, all elements of the form γ_S^μ for $S \subset P$ satisfy the conditions of Proposition 4.1.4 with $\mathrm{q}(\gamma_S^\mu) = \mathrm{q}(\gamma) = \mathrm{q}(\mu) \bmod 1$. Hence, we have for $r \geq 1$

$$\langle T(p^{2r}) f, v_{\gamma,\mu^\gamma_{\{p\}},P\setminus\{p\}} \rangle = T^{\mathrm{q}(\gamma)}(p^{2r}) \langle f, v_{p^r\gamma,p^r\mu^\gamma_{\{p\}},P\setminus\{p\}} \rangle \text{ and}$$
$$\langle T(p^{2r}) f, v_{\gamma^\mu_{\{p\}},\mu,P\setminus\{p\}} \rangle = T^{\mathrm{q}(\gamma)}(p^{2r}) \langle f, v_{p^r\gamma^\mu_{\{p\}},p^r\mu,P\setminus\{p\}} \rangle.$$

Furthermore, $p\gamma = p\gamma^{\mu}_{\{p\}}$ and $p\mu = p\mu^{\gamma}_{\{p\}}$, so that the right-hand sides are equal, thus

$$\begin{aligned}
&\langle \left(\sum_{r=0}^{\infty} \frac{T(p^{2r})}{p^{rs}} \right) f, v_{\gamma,\mu,P} \rangle \\
&\qquad = \langle f, v_{\gamma,\mu,P} \rangle + \langle \left(\sum_{r=1}^{\infty} \frac{T(p^{2r})}{p^{rs}} \right) f, v_{\gamma,\mu^{\gamma}_{\{p\}},P\setminus\{p\}} - v_{\gamma^{\mu}_{\{p\}},\mu,P\setminus\{p\}} \rangle \\
&\qquad = \langle f, v_{\gamma,\mu,P} \rangle + \sum_{r=1}^{\infty} \frac{T^{\mathfrak{q}(\gamma)}(p^{2r})}{p^{rs}} \langle f, v_{p^r\gamma,p^r\mu^{\gamma}_{\{p\}},P\setminus\{p\}} - v_{p^r\gamma^{\mu}_{\{p\}},p^r\mu,P\setminus\{p\}} \rangle \\
&\qquad = \langle f, v_{\gamma,\mu,P} \rangle.
\end{aligned}$$

□

Finally, we want to find a kernel function for the Hecke operators. We will need

Lemma 4.1.6. *Let $k \geq 2$ and $z \in \mathbb{H}$. Then for $x \in \mathbb{Q}$*

$$\sum_{n=-\infty}^{\infty} e(nx)(z+n)^{-k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{\substack{r \in \mathbb{Z}-x \\ r>0}} r^{k-1} e(rz).$$

Proof. The case $x \in \mathbb{Z}$ is well-known (see e.g. [49, (7.1.9)]). For $x = \frac{p}{q}$ with $(p, q) = 1$ write

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} e(nx)(z+n)^{-k} &= \sum_{m=0}^{q-1} \sum_{n=-\infty}^{\infty} e((m+qn)x)(z+m+qn)^{-k} \\
&= \sum_{m=0}^{q-1} e(mx) \frac{1}{q^k} \sum_{n=-\infty}^{\infty} \left(\frac{z+m}{q} + n \right)^{-k}.
\end{aligned}$$

Applying the equation for $x \in \mathbb{Z}$ with z replaced by $\frac{z+m}{q}$ we get

$$\begin{aligned}
&\sum_{m=0}^{q-1} e(mx) \frac{1}{q^k} \frac{(-2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1} e\left(r \frac{z+m}{q} \right) \\
&\qquad = \frac{(-2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} \left(\frac{r}{q} \right)^{k-1} e\left(\frac{r}{q} z \right) \frac{1}{q} \sum_{m=0}^{q-1} e\left(m\left(x + \frac{r}{q} \right) \right) \\
&\qquad = \frac{(-2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} \left(\frac{r}{q} \right)^{k-1} e\left(\frac{r}{q} z \right) \begin{cases} 1 & \text{if } r = -p \bmod q \\ 0 & \text{else} \end{cases} \\
&\qquad = \frac{(-2\pi i)^k}{(k-1)!} \sum_{\substack{r \in \mathbb{Z}-x \\ r>0}} r^{k-1} e(rz).
\end{aligned}$$

□

For $l \in \mathbb{Z}_{>0}$ we define functions $\omega_l : \mathbb{H} \times \mathbb{H} \to \mathbb{C}[D] \otimes \mathbb{C}[D]$ by

$$\omega_l(z,z') := \sum_{\gamma \in D} \sum_{\substack{a,b,c,d \in \mathbb{Z} \\ ad-bc=l^2 \\ (a,b,c,d)=1}} \frac{1}{(czz'+az+dz'+b)^k} \left(\rho_D^{(1)} \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)^{-1} e^{\gamma} \right) \otimes e^{\gamma}.$$

These are cusp forms of weight k for the Weil representation in both z and z'. Finally, we also define $\langle \cdot, \cdot \rangle : \mathbb{C}[D] \times (\mathbb{C}[D] \otimes \mathbb{C}[D]) \to \mathbb{C}[D]$ by

$$\langle v, w \otimes u \rangle := \langle v, w \rangle \cdot \overline{u}$$

for elements $v, w, u \in \mathbb{C}[D]$ and extend antilinearly in the second argument. The following proposition adapts [72, Proposition 1], which is originally due to Petersson, to the vector-valued case.

Proposition 4.1.7. *Let*

$$C(k) := \frac{i^k \pi}{2^{k-3}(k-1)}. \tag{4.1.3}$$

For $l \in \mathbb{Z}_{>0}$ the function $\overline{C(k)}^{-1} l^{2k-2} \omega_l(z, -\overline{z'})$ is the kernel function for the Hecke operator $T(l^2)$, i.e.

$$C(k)^{-1} l^{2k-2} \int_{\Gamma \backslash \mathbb{H}} \langle f(z), \omega_l(z, -\overline{z'}) \rangle y^k \frac{\mathrm{d}x \mathrm{d}y}{y^2} = (T(l^2) f)(z').$$

Proof. First note that we can write

$$\begin{aligned} \omega_l(z,z') &= \sum_{\gamma \in D} \sum_{\substack{a,b,c,d \in \mathbb{Z} \\ ad-bc=l^2 \\ (a,b,c,d)=1}} \frac{(cz+d)^{-k}}{(z' + \frac{az+b}{cz+d})^k} \rho_D^{(1)} \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \right) e^{\gamma} \otimes e^{\gamma} \\ &= l^{-k} \sum_{\gamma \in D} \sum_{M \in M_l} \frac{1}{(z' + \cdot)^k} \Big|_k [M](z) \rho_D^{(1)}(M)^{-1} e^{\gamma} \otimes e^{\gamma}. \end{aligned}$$

It is easily seen that $T(l^2)\omega_1(\cdot, z') = l^{2k-2}\omega_l(\cdot, z')$. Hence, since the Hecke operators are self-adjoint, it suffices to prove the proposition for $l = 1$. We set $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and

$\Gamma_\infty^+ = \langle T \rangle$. Then

$$\begin{aligned}
\omega_1(z,z') &= \sum_{\gamma \in D} \sum_{M \in \Gamma} \frac{1}{(z'+\cdot)^k} \Big|_k [M](z) \rho_D(M)^{-1} e^\gamma \otimes e^\gamma \\
&= \sum_{\gamma \in D} \sum_{M \in \Gamma_\infty^+ \backslash \Gamma} \sum_{n=-\infty}^{\infty} \frac{1}{(z'+\cdot)^k} \Big|_k [T^n M](z) \rho_D(T^n M)^{-1} e^\gamma \otimes e^\gamma \\
&= \sum_{\gamma \in D} \sum_{M \in \Gamma_\infty^+ \backslash \Gamma} \sum_{n=-\infty}^{\infty} \frac{e(-n\,\mathrm{q}(\gamma))}{(z'+\cdot+n)^k} \Big|_k [M](z) \rho_D(M)^{-1} e^\gamma \otimes e^\gamma \\
&= \sum_{\gamma \in D} \sum_{\substack{M \in \Gamma_\infty^+ \backslash \Gamma \\ M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}} (cz+d)^{-k} \sum_{n=-\infty}^{\infty} \frac{e(-n\,\mathrm{q}(\gamma))}{(z'+Mz+n)^k} \rho_D(M^{-1}) e^\gamma \otimes e^\gamma.
\end{aligned}$$

By Lemma 4.1.6 this equals

$$\begin{aligned}
&\sum_{\gamma \in D} \sum_{\substack{M \in \Gamma_\infty^+ \backslash \Gamma \\ M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}} (cz+d)^{-k} \frac{(-2\pi i)^k}{(k-1)!} \sum_{\substack{r \in \mathbb{Z}+\mathrm{q}(\gamma) \\ r>0}} r^{k-1} e(r(z'+Mz)) \rho_D(M^{-1}) e^\gamma \otimes e^\gamma \\
&= \frac{(-2\pi i)^k}{(k-1)!} \sum_{\gamma \in D} \sum_{\substack{r \in \mathbb{Z}+\mathrm{q}(\gamma) \\ r>0}} r^{k-1} \left(\sum_{\substack{M \in \Gamma_\infty^+ \backslash \Gamma \\ M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}} (cz+d)^{-k} e(rMz) \rho_D(M^{-1}) e^\gamma \right) \otimes e(rz') e^\gamma \\
&= \frac{(-2\pi i)^k}{(k-1)!} \sum_{\gamma \in D} \sum_{\substack{r \in \mathbb{Z}+\mathrm{q}(\gamma) \\ r>0}} r^{k-1} P_{k,D,\gamma,r}(z) \otimes e(rz') e^\gamma,
\end{aligned}$$

where $P_{k,D,\gamma,r}$ is the Poincaré series of index (γ, r) defined in Section 1.3. In [11] it is shown that

$$(f, P_{k,D,\gamma,r}) = 2 \frac{(k-2)!}{(4\pi r)^{k-1}} c(\gamma, r)$$

for a cusp form

$$f(\tau) = \sum_{\beta \in D} \sum_{\substack{r \in \mathbb{Z}+\mathrm{q}(\gamma) \\ r>0}} c(\beta, r) e(r\tau) e^\beta.$$

We thus have

$$\begin{aligned}
C(k)^{-1}(f, \omega_1(z, -\overline{z'})) &= C(k)^{-1} \frac{(2\pi i)^k}{(k-1)!} \sum_{\gamma \in D} \sum_{\substack{r \in \mathbb{Z}+\mathrm{q}(\gamma) \\ r>0}} r^{k-1} (f, P_{\gamma,r}) e(rz') e^\gamma \\
&= C(k)^{-1} 2 \frac{(2\pi i)^k}{(k-1)!} \frac{(k-2)!}{(4\pi)^{k-1}} \sum_{\gamma \in D} \sum_{\substack{r \in \mathbb{Z}+\mathrm{q}(\gamma) \\ r>0}} c(\gamma, r) e(r\tau) e^\gamma \\
&= f(z').
\end{aligned}$$

□

4.2 Vector-valued Eisenstein series

In this section we will study a relation between the Eisenstein series for genus $n = 2$ and the Hecke operators for $n = 1$. In the scalar-valued case a similar result was shown in [34] and [4], called the pullback formula. We generalize the pullback formula to the vector-valued case, however using a different approach. In particular, we will prove that $\partial_h E^{(2)}_{m/2}\left(\left(\begin{smallmatrix} z & 0 \\ 0 & z' \end{smallmatrix}\right)\right)$ is the sum of the kernel functions for the Hecke operators from the previous section. When $h = 0$ we get, as an additional term, the product of the genus 1 Eisenstein series in z and in z' (also cf. [67]).

For a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Mat}_2(\mathbb{Z})$, we introduce the notation $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)' = \left(\begin{smallmatrix} d & b \\ c & a \end{smallmatrix}\right)$ and $\gcd\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \gcd(a, b, c, d)$.

Proposition 4.2.1. *Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^{(2)}$ and denote by C_1, C_2, D_1 and D_2 the first and second column of C and D respectively. We define the maps*

$$\varphi : \Gamma^{(2)} \to \mathrm{Mat}_2(\mathbb{Z}), \quad \varphi(M) = \begin{pmatrix} \det(C_1, D_2) & \det(D) \\ \det(C) & \det(D_1, C_2) \end{pmatrix}$$

and

$$\nu : \Gamma^{(2)} \to \mathbb{Z}, \quad \nu(M) = \det(C_1, D_1).$$

Then we have

$$\begin{aligned} \varphi(M \cdot u(A)) &= \varphi(M) \cdot A, \\ \varphi(M \cdot d(A)) &= A' \cdot \varphi(M), \\ \nu(M \cdot u(A)) &= \nu(M), \\ \nu(M \cdot d(A)) &= \nu(M) \end{aligned}$$

for $M \in \Gamma^{(2)}$ and $A \in \Gamma^{(1)}$. The map

$$\begin{aligned} \phi : \Gamma^{(2)}_\infty \backslash \Gamma^{(2)} &\to \{(\delta, l) \in \mathrm{Mat}_2(\mathbb{Z}) \times \mathbb{Z} \mid \det(\delta) = l^2, \gcd(\delta) = 1\} / \{(I, 1), (-I, -1)\}, \\ M &\mapsto (\varphi(M), \nu(M)) \end{aligned}$$

is bijective.

Proof. The first relations follow from simple computations. It remains to prove that ϕ is a bijection. First we show that it is well-defined, i.e. that $\det(\varphi(M)) = \det(C_1, D_1)^2$, that $\gcd(\varphi(M)) = 1$ and that for any $\tilde{M} \in \Gamma^{(2)}_\infty$ the matrices $\tilde{M}M$ and

M have the same image under ϕ. The last statement follows immediately from the fact that any $\tilde{M} \in \Gamma_\infty^{(2)}$ is of the form

$$\begin{pmatrix} U & B \\ 0 & (U^T)^{-1} \end{pmatrix}$$

for some $U \in \mathrm{GL}_2(\mathbb{Z})$ and that $\det(U) = \pm 1$. Now let $C = \left(\begin{smallmatrix} c_1 & c_2 \\ c_3 & c_4 \end{smallmatrix}\right)$ and $D = \left(\begin{smallmatrix} d_1 & d_2 \\ d_3 & d_4 \end{smallmatrix}\right)$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \varphi(M) = \begin{pmatrix} c_1d_4 - d_2c_3 & d_1d_4 - d_2d_3 \\ c_1c_4 - c_2c_3 & d_1c_4 - c_2d_3 \end{pmatrix}$$

and so

$$\begin{aligned} \det(\varphi(M)) &= ad - bc \\ &= (c_1d_4 - d_2c_3)(d_1c_4 - c_2d_3) - (d_1d_4 - d_2d_3)(c_1c_4 - c_2c_3) \\ &= -c_1c_2d_3d_4 + c_1c_4d_2d_3 + c_2c_3d_1d_4 - c_3c_4d_1d_2. \end{aligned}$$

Because M is symplectic, we know that $CD^T = DC^T$, which is equivalent to $c_1d_3 - d_1c_3 = d_2c_4 - c_2d_4$. Therefore, we have

$$\begin{aligned} \det(C_1, D_1)^2 &= (c_1d_3 - d_1c_3)(d_2c_4 - c_2d_4) \\ &= -c_1c_2d_3d_4 + c_1c_4d_2d_3 + c_2c_3d_1d_4 - c_3c_4d_1d_2 \\ &= \det(\varphi(M)). \end{aligned}$$

By Lemma 4.1.1 every $\tilde{M} \in \mathrm{Mat}_2(\mathbb{Z})$ of determinant l^2 can be written as $A \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \delta \end{smallmatrix}\right) B$ for $A, B \in \Gamma$ and $\alpha\delta = l^2$ and we have

$$\varphi(M \cdot d(A') \cdot u(B)) = A\varphi(M)B. \tag{4.2.1}$$

We show that there exists an $M \in \Gamma^{(2)}$ such that $\varphi(M) = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \delta \end{smallmatrix}\right)$ if and only if $(\alpha, \delta) = 1$:

It is well known that a right coset decomposition of the integral 2×2 matrices of determinant c is given by

$$\bigcup_{\substack{c_1 > 0 \\ c_1c_4 = c \\ c_2 \bmod c_4}} \Gamma \begin{pmatrix} c_1 & c_2 \\ 0 & c_4 \end{pmatrix}.$$

Hence, if $\varphi(M) = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \delta \end{smallmatrix}\right)$ for $M \in \Gamma_\infty^{(2)} \backslash \Gamma^{(2)}$, we can choose the representative M such that $C = \left(\begin{smallmatrix} c_1 & c_2 \\ 0 & 0 \end{smallmatrix}\right)$. We then find that $c_1d_4 = \alpha$ and $-c_2d_3 = \delta$ and $c_1d_3 = -c_2d_4 = l$. Therefore, $c_1 \mid \gcd(\alpha, l)$. If $x = \gcd(\alpha, l)/c_1$, then $x \mid d_4$ and $x \mid d_3$. But the last row of M is $(0,0,d_3,d_4)$ so that $x=1$ and $c_1=\gcd(\alpha,l)$. By an analogous argument $c_2=\pm\gcd(\delta,l)$. Using the Laplace expansion along the 3rd row, we find that

$$\begin{aligned}1=\det(M)&=c_1\cdot\det(\ldots)-c_2\cdot\det(\ldots)+\det(D)\cdot\det(A)\\&=c_1\cdot\det(\ldots)-c_2\cdot\det(\ldots),\end{aligned}$$

so that $1=\gcd(c_1,c_2)=\gcd(\gcd(\alpha,l),\gcd(\delta,l))=\gcd(\alpha,\delta)$. Therefore, ϕ is well-defined.
Recall the definition of

$$\mathcal{A}_l=\begin{pmatrix}l^2+l & -l-1 & -1 & -l-1\\ -l-1 & 1 & 0 & 0\\ -l & 1 & 0 & 0\\ 0 & 0 & -1 & -l\end{pmatrix}\in\Gamma^{(2)}.$$

We have

$$\varphi(\mathcal{A}_l)=\begin{pmatrix}l^2 & 0\\ 0 & 1\end{pmatrix}$$

and $\nu(\mathcal{A}_l)=l$. Using (4.2.1) and Lemma 4.1.1 this implies that ϕ is surjective. For injectivity recall that if $\varphi(M)=\left(\begin{smallmatrix}l^2&0\\0&1\end{smallmatrix}\right)$, then we can assume that $C=\left(\begin{smallmatrix}c_1&c_2\\0&0\end{smallmatrix}\right)$ with $c_1=\gcd(l^2,l)=l$ and $c_2=\pm\gcd(1,l)=\pm1$. Then $d_4=l$, $d_3=\mp1$ and $d_2=\mp ld_1$. Now

$$\begin{aligned}\begin{pmatrix}1&b\\0&1\end{pmatrix}\cdot(C,D)&=\begin{pmatrix}1&b\\0&1\end{pmatrix}\cdot\left(\begin{pmatrix}l&\pm1\\0&0\end{pmatrix},\begin{pmatrix}d_1&\mp ld_1\\ \mp1&l\end{pmatrix}\right)\\&=\left(\begin{pmatrix}l&\pm1\\0&0\end{pmatrix},\begin{pmatrix}d_1\mp b&\mp l(d_1\mp b)\\ \mp1&l\end{pmatrix}\right).\end{aligned}$$

Therefore, for every d_1 the corresponding matrices M are in the same coset mod $\Gamma_\infty^{(2)}$. We choose a representative with $d_1=0$. Then $\nu(M)=l$ if and only if $\pm=-$. So there exists exactly one pair $(C,D)\in\mathrm{GL}_2(\mathbb{Z})\backslash(\mathrm{Mat}_2(\mathbb{Z})\times\mathrm{Mat}_2(\mathbb{Z}))$ such that for $M=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)$ we have $\varphi(M)=\left(\begin{smallmatrix}l^2&0\\0&1\end{smallmatrix}\right)$ and $\nu(M)=l$. If $M,\tilde{M}\in\Gamma^{(2)}$ have identical C and D, then $\tilde{M}M^{-1}\in\Gamma_\infty^{(2)}$ and so ϕ is injective. □

For the next proposition we will need

Lemma 4.2.2. *Let D be a discriminant form of even signature, $l\in\mathbb{Z}$, $A\in M_l$ and $B\in\mathrm{SL}_2(\mathbb{Z})$. Then $(AB)'=B'A'$ and*

$$\sum_{\gamma\in D}\rho_D(A)^{-1}e^\gamma\otimes\rho_D(B)^{-1}e^\gamma=\sum_{\gamma\in D}\rho_D(B'A)^{-1}e^\gamma\otimes e^\gamma.$$

Proof. The fact that $(AB)' = B'A'$ follows from a simple calculation. Now let $B_1, B_2 \in \mathrm{SL}_2(\mathbb{Z})$ and assume that the identity holds whenever B is equal to B_1 or B_2. Then for $A \in M_l$ by Proposition 1.4.1 also

$$\begin{aligned}\sum_{\gamma\in D} \rho_D(A)^{-1}e^\gamma \otimes \rho_D(B_1B_2)^{-1}e^\gamma &= \rho_D^{(2)}(d(B_2^{-1}))\sum_{\gamma\in D}\rho_D(A)^{-1}e^\gamma\otimes\rho_D(B_1)^{-1}e^\gamma\\ &= \rho_D^{(2)}(d(B_2^{-1}))\sum_{\gamma\in D}\rho_D(B_1'A)^{-1}e^\gamma\otimes e^\gamma\\ &= \sum_{\gamma\in D}\rho_D(B_1'A)^{-1}e^\gamma\otimes\rho_D(B_2)^{-1}e^\gamma\\ &= \sum_{\gamma\in D}\rho_D(B_2'B_1'A)^{-1}e^\gamma\otimes e^\gamma\\ &= \sum_{\gamma\in D}\rho_D((B_1B_2)'A)^{-1}e^\gamma\otimes e^\gamma,\end{aligned}$$

so it suffices to prove the identity for B equal to generators of $\mathrm{SL}_2(\mathbb{Z})$, i.e. $J_1 = \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ and $n(1) = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Note that both $J_1' = J_1$ and $n(1)' = n(1)$. For $n(1)$ the identity is trivial, while for J_1 we have

$$\begin{aligned}\sum_{\gamma\in D}\rho_D(A)^{-1}e^\gamma\otimes\rho_D(J_1)^{-1}e^\gamma &= \sum_{\gamma\in D}\rho_D(A)^{-1}e^\gamma\otimes\frac{e(-\operatorname{sign}(D)/8)}{|D|}\sum_{\beta\in D}e(-(\beta,\gamma))e^\beta\\ &= \sum_{\beta\in D}\rho_D(A)^{-1}\frac{e(-\operatorname{sign}(D)/8)}{|D|}\sum_{\gamma\in D}e(-(\beta,\gamma))e^\gamma\otimes e^\beta\\ &= \sum_{\beta\in D}\rho_D(A)^{-1}\rho_D(J_1)^{-1}e^\beta\otimes e^\beta.\end{aligned}$$

□

We can now show that in a special case, the action of an element $M \in \Gamma^{(2)}$ in the Weil representation is given in terms of the action of $\varphi(M)$.

Proposition 4.2.3. *Let D be a discriminant form of even signature. Let $M \in \Gamma^{(2)}$ and $\epsilon = \mathrm{sgn}(\nu(M))$ with $\mathrm{sgn}(0) = -1$. Then*

$$\rho_D^{(2)}(M)^{-1}(e^0\otimes e^0) = \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}}\sum_{\gamma\in D}\rho_D(\varphi(M))^{-1}e^{-\epsilon\gamma}\otimes e^\gamma.$$

Proof. We first show the statement for $M = \mathcal{A}_l = Jn_1Jn_2Jn_3$ with

$$n_1 = n\left(\begin{pmatrix}0 & -1\\ -1 & -l\end{pmatrix}\right),\quad n_2 = n\left(\begin{pmatrix}l^2+l & -l-1\\ -l-1 & 1\end{pmatrix}\right),\quad n_3 = n\left(\begin{pmatrix}0&0\\0&1\end{pmatrix}\right).$$

Then we find

$$\begin{aligned}
&\rho_D^{(2)}(\mathcal{A}_l)^{-1}(e^0 \otimes e^0) \\
&\qquad = \rho_D^{(2)}(n_1 J n_2 J n_3)^{-1} \frac{e(-\operatorname{sign}(D)/4)}{|D|} \sum_{(\mu_1,\mu_2)\in D^2} e^{\mu_1} \otimes e^{\mu_2} \\
&\qquad = \rho_D^{(2)}(J n_2 J n_3)^{-1} \frac{e(-\operatorname{sign}(D)/4)}{|D|} \sum_{(\mu_1,\mu_2)\in D^2} e((\mu_1,\mu_2) + l\,\mathrm{q}(\mu_2)) e^{\mu_1} \otimes e^{\mu_2} \\
&\qquad = \rho_D^{(2)}(n_2 J n_3)^{-1} \frac{e(-\operatorname{sign}(D)/2)}{|D|^2} \sum_{(\beta_1,\beta_2)\in D^2} \sum_{(\mu_1,\mu_2)\in D^2} e(-(\mu_1,\beta_1) - (\mu_2,\beta_2)) \\
&\qquad\qquad\quad e((\mu_1,\mu_2) + l\,\mathrm{q}(\mu_2)) e^{\beta_1} \otimes e^{\beta_2} \\
&\qquad = \rho_D^{(2)}(n_2 J n_3)^{-1} \frac{e(-\operatorname{sign}(D)/2)}{|D|} \sum_{(\beta_1,\beta_2)\in D^2} e(-(\beta_1,\beta_2) + l\,\mathrm{q}(\beta_1)) e^{\beta_1} \otimes e^{\beta_2},
\end{aligned}$$

where we used that

$$\sum_{\mu_1\in D} e((\mu_1, \mu_2 - \beta_1)) = \begin{cases} |D| & \text{if } \mu_2 = \beta_1 \\ 0 & \text{otherwise.} \end{cases}$$

We proceed

$$\begin{aligned}
&\rho_D^{(2)}(J n_3)^{-1} \frac{e(-\operatorname{sign}(D)/2)}{|D|} \sum_{(\beta_1,\beta_2)\in D^2} e(-(l^2+l)\,\mathrm{q}(\beta_1) + (l+1)(\beta_1,\beta_2) - \mathrm{q}(\beta_2)) \\
&\qquad\qquad e(-(\beta_1,\beta_2) + l\,\mathrm{q}(\beta_1)) e^{\beta_1} \otimes e^{\beta_2} \\
&\quad = \rho_D^{(2)}(n_3)^{-1} \frac{e(-3\operatorname{sign}(D)/4)}{|D|^2} \sum_{(\gamma_1,\gamma_2)\in D^2} \sum_{(\beta_1,\beta_2)\in D^2} e(-(\beta_1,\gamma_1) - (\beta_2,\gamma_2)) \\
&\qquad\qquad e(-\mathrm{q}(l\beta_1) + l(\beta_1,\beta_2) - \mathrm{q}(\beta_2)) e^{\gamma_1} \otimes e^{\gamma_2} \\
&\quad = \frac{e(-3\operatorname{sign}(D)/4)}{|D|^2} \sum_{(\gamma_1,\gamma_2)\in D^2} \sum_{(\beta_1,\beta_2)\in D^2} e(-\mathrm{q}(\beta_2 - l\beta_1 + \gamma_2)) \\
&\qquad\qquad e(-(\gamma_1 + l\gamma_2, \beta_1)) e^{\gamma_1} \otimes e^{\gamma_2}.
\end{aligned}$$

Taking the sum over β_2 and using Milgram's formula we get

$$\begin{aligned}
&\frac{e(\operatorname{sign}(D)/8)}{|D|^{\frac{3}{2}}} \sum_{(\gamma_1,\gamma_2)\in D^2} \sum_{\beta_1\in D} e(-(\gamma_1 + l\gamma_2, \beta_1)) e^{\gamma_1} \otimes e^{\gamma_2} \\
&\qquad = \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\gamma_2\in D} e^{-l\gamma_2} \otimes e^{\gamma_2} \\
&\qquad = \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\gamma_2\in D} \rho_D\left(\begin{pmatrix} l^2 & 0 \\ 0 & 1 \end{pmatrix}\right)^{-1} e^{-\epsilon\gamma_2} \otimes e^{\gamma_2}.
\end{aligned}$$

(Note that $\rho_D\left(\left(\begin{smallmatrix} l^2 & 0\\ 0 & 1\end{smallmatrix}\right)\right)^{-1} e^\gamma = e^{|l|\gamma}$ by definition.)
Now let $M \in \Gamma^{(2)}$ be arbitrary. By Proposition 4.2.1 and Lemma 4.1.1 we have

$$M \in \Gamma^{(2)}_\infty \cdot \mathcal{A}_l u(A)d(B)$$

for some $l \in \mathbb{Z}$ and $A, B \in \Gamma^{(1)}$. Since $\tilde{M} = n(S)a(U) \in \Gamma^{(2)}_\infty$ acts on $e^0 \otimes e^0$ as multiplication by $\det(U)^{\operatorname{sign}(D)/2}$ and $\varphi(\tilde{M}M) = \det(U)\varphi(M)$ and $\nu(\tilde{M}M) = \det(U)\nu(M)$, we can assume that M is equal to $\mathcal{A}_l u(A)d(B)$. By Proposition 1.4.1 and Lemma 4.2.2 we have

$$\begin{aligned}
&\rho_D^{(2)}(\mathcal{A}_l u(A)d(B))^{-1}(e^0 \otimes e^0)\\
&\qquad = \rho_D^{(2)}(u(A)\cdot d(B))^{-1}\rho_D^{(2)}(\mathcal{A}_l)^{-1}(e^0 \otimes e^0)\\
&\qquad = \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\gamma\in D} \rho_D(A)^{-1}\rho_D(\varphi(\mathcal{A}_l))^{-1}e^{-\epsilon\gamma} \otimes \rho_D(B)^{-1}e^\gamma\\
&\qquad = \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\gamma\in D} \rho_D(B'\varphi(\mathcal{A}_l)A)^{-1}e^{-\epsilon\gamma} \otimes e^\gamma\\
&\qquad = \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\gamma\in D} \rho_D(\varphi(\mathcal{A}_l u(A)d(B)))^{-1}e^{-\epsilon\gamma} \otimes e^\gamma.
\end{aligned}$$

□

Recall that $M_l = \{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \operatorname{Mat}_2(\mathbb{Z}) \mid ad - bc = l^2, \gcd(a,b,c,d) = 1\}$. For the case $l = 0$ we will need

Lemma 4.2.4. *We define a function $\psi : \Gamma^{(1)} \times \Gamma^{(1)} \to M_0$ by*

$$\left(\begin{pmatrix} * & * \\ r & s\end{pmatrix}, \begin{pmatrix} * & * \\ t & u\end{pmatrix}\right) \mapsto \begin{pmatrix} ru & su \\ rt & st\end{pmatrix}.$$

Then ψ defines a bijection between $\Gamma^{(1)}_\infty\backslash\Gamma^{(1)} \times \Gamma^{(1)}_\infty\backslash\Gamma^{(1)}$ and $M_0/\{\pm 1\}$. Let D be a discriminant form of even signature, then we have

$$\rho_D(A)^{-1}e^0 \otimes \rho_D(B)^{-1}e^0 = \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\gamma\in D} \rho_D(\psi(A,B))^{-1}e^\gamma \otimes e^\gamma.$$

Proof. It is easy to see that ψ is well-defined as a mapping from $\Gamma^{(1)}_\infty\backslash\Gamma^{(1)} \times \Gamma^{(1)}_\infty\backslash\Gamma^{(1)}$ to $M_0/\{\pm 1\}$. A simple computation shows that for $A, B, C, D \in \Gamma$ we have

$$\psi(A\cdot C, B\cdot D) = D' \cdot \psi(A,B)\cdot C.$$

Noting that $\psi(I, -J_1) = \left(\begin{smallmatrix} 0 & 0\\ 0 & 1\end{smallmatrix}\right)$, surjectivity follows from Lemma 4.1.1. For injectivity assume that $\psi(A,B) = \psi(C,D)$ and so

$$\psi(AC^{-1}, BD^{-1}) = \psi(I,I) = \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}.$$

We need to show that $AC^{-1}, BD^{-1} \in \Gamma_\infty$:
Suppose that

$$AC^{-1} = \begin{pmatrix} * & * \\ r & s \end{pmatrix}, \quad BD^{-1} = \begin{pmatrix} * & * \\ t & u \end{pmatrix}.$$

Then $ru = 0$ and $su = 1$, so that we must have $r = 0$. Because $st = 0$ and $su = 1$, we also have $t = 0$ and $s = u = \pm 1$, i.e. $AC^{-1}, BD^{-1} \in \Gamma_\infty$.
Now we find that

$$\begin{aligned} \rho_D(I)^{-1}e^0 \otimes \rho_D(-J_1)^{-1}e^0 &= \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\gamma \in D} e^0 \otimes e^\gamma \\ &= \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\gamma \in D} \rho_D\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)^{-1} e^\gamma \otimes e^\gamma. \end{aligned}$$

For arbitrary $A, B \in \Gamma$ we once again use Proposition 1.4.1 and Lemma 4.2.2 to get

$$\begin{aligned} &\rho_D(A)^{-1}e^0 \otimes \rho_D(B)^{-1}e^0 \\ &\quad = \rho_D^{(2)}(u(A)d(J_1B))^{-1}(\rho_D(I)^{-1}e^0 \otimes \rho_D(-J_1)^{-1}e^0) \\ &\quad = \rho_D^{(2)}(u(A)d(J_1B))^{-1}\left(\frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\gamma \in D} \rho_D(\psi(I, -J_1))^{-1}e^\gamma \otimes e^\gamma\right) \\ &\quad = \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\gamma \in D} \rho_D(A)^{-1}\rho_D(\psi(I, -J_1))^{-1}e^\gamma \otimes \rho_D(J_1B)^{-1}e^\gamma \\ &\quad = \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\gamma \in D} \rho_D(B'J_1\psi(I, -J_1)A)^{-1}e^\gamma \otimes e^\gamma \\ &\quad = \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\gamma \in D} \rho_D(\psi(A, B))^{-1}e^\gamma \otimes e^\gamma. \end{aligned}$$

□

The next result finally describes the relation between $E_k^{(2)}$ and the Hecke operators and is due to Stein (cf. [67, Theorem 5.3]). We include it together with its proof because we will generalize the argument to establish Theorem 4.2.6. Recall that the function ω_l defined on page 112 is the kernel function of the Hecke operator $T(l^2)$.

Theorem 4.2.5. *Let D be a discriminant form of even signature and let $k > 3$ with $k = \operatorname{sign}(D)/2 \bmod 4$. Then we have*

$$E_k^{(2)}\left(\left(\begin{smallmatrix} z & 0 \\ 0 & z' \end{smallmatrix}\right)\right) = E_k^{(1)}(z) \otimes E_k^{(1)}(z') + \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{l \in \mathbb{Z}_{>0}} \omega_l(z, z').$$

Proof. Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^{(2)}$ and consider $C = \left(\begin{smallmatrix} c_1 & c_2 \\ c_3 & c_4 \end{smallmatrix}\right)$ and $D = \left(\begin{smallmatrix} d_1 & d_2 \\ d_3 & d_4 \end{smallmatrix}\right)$. Then for $Z = \left(\begin{smallmatrix} z & 0 \\ 0 & z' \end{smallmatrix}\right)$

$$\begin{aligned}\det(CZ+D) &= \det\left(\begin{pmatrix} c_1 z + d_1 & c_2 z' + d_2 \\ c_3 z + d_3 & c_4 z' + d_4 \end{pmatrix}\right)\\ &= (c_1 z + d_1)(c_4 z' + d_4) - (c_2 z' + d_2)(c_3 z + d_3)\\ &= (c_1c_4 - c_2c_3) zz' + (c_1 d_4 - d_2 c_3) z + (d_1 c_4 - c_2 d_3) z' + (d_1 d_4 - d_2 d_3)\\ &= czz' + az + dz' + b,\end{aligned}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \varphi(M)$. Hence, $E_k^{(2)}(\left(\begin{smallmatrix} z & 0 \\ 0 & z' \end{smallmatrix}\right))$ is equal to

$$\begin{aligned}&\sum_{\substack{M \in \Gamma_\infty^{(2)} \backslash \Gamma^{(2)} \\ \varphi(M) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}} \frac{1}{(czz' + az + dz' + b)^k} \rho_D^{(2)}(M)^{-1} (e^0 \otimes e^0)\\ &= \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\substack{M \in \Gamma_\infty^{(2)} \backslash \Gamma^{(2)} \\ \varphi(M) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}} \frac{1}{(czz' + az + dz' + b)^k} \sum_{\gamma \in D} \rho_D(\varphi(M))^{-1} e^{-\operatorname{sgn}(\nu(M))\gamma} \otimes e^\gamma\\ &= \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{l \in \mathbb{Z}_{>0}} \sum_{M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_l} \frac{1}{(czz' + az + dz' + b)^k} \sum_{\gamma \in D} \rho_D(M)^{-1} e^\gamma \otimes e^\gamma\\ &\quad + \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_0/\{\pm 1\}} \frac{1}{(czz' + az + dz' + b)^k} \sum_{\gamma \in D} \rho_D(M)^{-1} e^\gamma \otimes e^\gamma,\end{aligned}$$

where we applied Propositions 4.2.1 and 4.2.3 as well as the fact that

$$\begin{aligned}&\frac{1}{(-czz' - az - dz' - b)^k} \sum_{\gamma \in D} \rho_D(-\varphi(M))^{-1} e^{-\gamma} \otimes e^\gamma\\ &\qquad = \frac{(-1)^k}{(czz' + az + dz' + b)^k} \sum_{\gamma \in D} \rho_D(\varphi(M))^{-1} \rho_D(-I) e^{-\gamma} \otimes e^\gamma\\ &\qquad = \frac{(-1)^k e(\operatorname{sign}(D)/4)}{(czz' + az + dz' + b)^k} \sum_{\gamma \in D} \rho_D(\varphi(M))^{-1} e^\gamma \otimes e^\gamma\end{aligned}$$

and $e(\operatorname{sign}(D)/4) = i^{\operatorname{sign}(D)} = i^{2k} = (-1)^k$. So it remains to show that the term for

$l = 0$ is equal to $E_k^{(1)}(z) \otimes E_k^{(1)}(z')$. In fact, by Lemma 4.2.4, we have

$$\begin{aligned}
&\frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{M \in M_0/\{\pm 1\}} \frac{1}{(czz' + az + dz' + b)^k} \sum_{\gamma \in D} \rho_D(M)^{-1} e^\gamma \otimes e^\gamma \\
&= \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\substack{A,B \in \Gamma_\infty^{(1)} \backslash \Gamma^{(1)} \\ \psi(A,B) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}} \frac{1}{(czz' + az + dz' + b)^k} \sum_{\gamma \in D} \rho_D(\psi(A,B))^{-1} e^\gamma \otimes e^\gamma \\
&= \sum_{\substack{A,B \in \Gamma_\infty^{(1)} \backslash \Gamma^{(1)} \\ \psi(A,B) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}} \frac{1}{(czz' + az + dz' + b)^k} \rho_D(A)^{-1} e^0 \otimes \rho_D(B)^{-1} e^0 \\
&= \left(\sum_{\substack{A \in \Gamma_\infty^{(1)} \backslash \Gamma^{(1)} \\ A = \left(\begin{smallmatrix} * & * \\ r & s \end{smallmatrix}\right)}} \frac{1}{(rz + s)^k} \rho_D(A)^{-1} e^0 \right) \otimes \left(\sum_{\substack{B \in \Gamma_\infty^{(1)} \backslash \Gamma^{(1)} \\ B = \left(\begin{smallmatrix} * & * \\ t & u \end{smallmatrix}\right)}} \frac{1}{(tz' + u)^k} \rho_D(B)^{-1} e^0 \right) \\
&= E_k^{(1)}(z) \otimes E_k^{(1)}(z').
\end{aligned}$$

□

Recall the differential operator ∂_h defined on page 40. Applying it to the Eisenstein series we obtain

Theorem 4.2.6. *Let D be a discriminant form of even signature, $m > 6$ with $m = \operatorname{sign}(D) \bmod 8$ and $k = m/2 + h$ with $h \geq 0$. Then*

$$(\partial_h E_{m/2}^{(2)})(z, z') = G_m^h(1,1) \cdot \frac{(k-1)!}{(m/2-1)!} \cdot \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{l \in \mathbb{Z}_{>0}} l^h \omega_l(z, z').$$

Proof. We have already seen that for any $M \in \Gamma_\infty^{(2)} \backslash \Gamma^{(2)}$ we can find a representative such that $M = \mathcal{A}_l u(A) d(B)$ for suitable $A, B \in \Gamma^{(1)}$ and $l \in \mathbb{Z}_{\geq 0}$. Hence,

$$E_{m/2}^{(2)}(Z) = \sum_{l \in \mathbb{Z}_{\geq 0}} \sum_{A,B} 1|_{m/2}[\mathcal{A}_l u(A) d(B)] \rho_D^{(2)}(M)^{-1}(e^0 \otimes e^0),$$

where the second sum ranges over the appropriate A and B (which depend on l). It was shown in [38, section 3.1.1] that ∂_h commutes with the slash operator of $u(A)$ and $d(B)$ on $\mathbb{H} \times \mathbb{H} \subset \mathbb{H}_2$, i.e. for a $f : \mathbb{H}_2 \to \mathbb{C}$ we have

$$\begin{aligned}
\partial_h(f|_{m/2}[u(A)]) &= (\partial_h f)|_{m/2+h}^1[A] \quad \text{and} \\
\partial_h(f|_{m/2}[d(B)]) &= (\partial_h f)|_{m/2+h}^2[B],
\end{aligned}$$

where $|_k^1$ acts on the first and $|_k^2$ on the second variable (recall that $(\partial_h f) : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$). Therefore, we obtain

$$\begin{aligned}(\partial_h E^{(2)}_{m/2})(z_1, z_4) &= \sum_{l \in \mathbb{Z}_{\geq 0}} \sum_{A,B} \partial_h (1|_{m/2}[\mathcal{A}_l a(A) d(B)] \rho_D^{(2)}(M)^{-1}(e^0 \otimes e^0) \\ &= \sum_{l \in \mathbb{Z}_{\geq 0}} \sum_{A,B} (\partial_h 1|_{m/2}[\mathcal{A}_l])|_k^1[A]|_k^2[B] \rho_D^{(2)}(M)^{-1}(e^0 \otimes e^0).\end{aligned}$$

Since

$$\mathcal{A}_l = \begin{pmatrix} l^2+l & -l-1 & -1 & -l-1 \\ -l-1 & 1 & 0 & 0 \\ -l & 1 & 0 & 0 \\ 0 & 0 & -1 & -l \end{pmatrix},$$

we have

$$1|_{m/2}[\mathcal{A}_l](Z) = \det\left(\begin{pmatrix} -lz_1 + z_2 & -lz_2 + z_4 \\ -1 & -l \end{pmatrix}\right)^{-m/2} = (l^2 z_1 - 2l z_2 + z_4)^{-m/2}.$$

Then

$$\frac{\partial}{\partial z_2}(l^2 z_1 - 2l z_2 + z_4)^{-s} = 2ls(l^2 z_1 - 2l z_2 + z_4)^{-s-1}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial z_1 \partial z_4}(l^2 z_1 - 2l z_2 + z_4)^{-s} &= \frac{\partial}{\partial z_4} l^2 (-s)(l^2 z_1 - 2l z_2 + z_4)^{-s-1} \\ &= l^2 s(s+1)(l^2 z_1 - 2l z_2 + z_4)^{-s-2}.\end{aligned}$$

Since $G_m^h(x, y^2)$ is homogeneous of degree h, we find

$$\begin{aligned}&G_m^h\left(\frac{1}{2}\frac{\partial}{\partial z_2}, \frac{\partial^2}{\partial z_1 \partial z_4}\right)(l^2 z_1 - 2l z_2 + z_4)^{-m/2} \\ &\qquad = G_m^h(l, l^2) \cdot m/2 \cdot \ldots \cdot (k-1) \cdot (l^2 z_1 - 2l z_2 + z_4)^{-k} \\ &\qquad = G_m^h(1,1) \cdot \frac{(k-1)!}{(m/2-1)!} \cdot l^h \cdot 1|_k[\mathcal{A}_l](Z).\end{aligned}$$

Hence,

$$\begin{aligned}
(\partial_h E^{(2)}_{m/2})(z_1, z_4) &= G^h_m(1,1) \cdot \frac{(k-1)!}{(m/2-1)!} \cdot \sum_{l \in \mathbb{Z}_{\geq 0}} l^h \\
&\qquad \sum_{A,B} 1|_k [\mathcal{A}_l u(A) d(B)] \left(\left(\begin{smallmatrix} z_1 & 0 \\ 0 & z_4 \end{smallmatrix} \right) \right) \rho_D^{(2)}(M)^{-1}(e^0 \otimes e^0) \\
&= G^h_m(1,1) \cdot \frac{(k-1)!}{(m/2-1)!} \cdot \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \cdot \sum_{l \in \mathbb{Z}_{>0}} l^h \\
&\qquad \sum_{M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in M_l} \frac{1}{(czz' + az + dz' + b)^k} \sum_{\gamma \in D} \rho_D(M)^{-1} e^\gamma \otimes e^\gamma \\
&= G^h_m(1,1) \cdot \frac{(k-1)!}{(m/2-1)!} \cdot \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{l \in \mathbb{Z}_{>0}} n^h \omega_l(z, z'),
\end{aligned}$$

where the last two steps are the same as in Theorem 4.2.5. □

4.3 The space of theta series

In this section we will prove the main theorem of this chapter. As explained in the introduction, we define a map from the space of cusp forms to the space generated by the theta series by integrating a cuspform against the genus theta series of Siegel genus 2 evaluated on a diagonal matrix. This process is called the doubling method. By the Siegel–Weil formula we can then substitute the genus theta series for the Eisenstein series. Using the results from Sections 4.1 and 4.2 we find that the resulting map is a linear combination of Hecke operators. Finally, we will show, that this map is bijective if the conditions of the main theorem are met.

Let D be a discriminant form of even signature and level N and let m be even with $m > p\text{-rank}(D)$ for all primes p. We set $G = II_{m,0}(D)$ and $k = m/2 + h$ with $h \geq 0$. By [55, Corollary 1.10.2] G is non-empty. We define a linear map

$$\Phi \coloneqq \Phi_D \coloneqq \Phi_{D,m,k} : \mathrm{S}_k(D) \to \Theta_{m,k}(D)$$

by

$$\Phi_{D,m,k}(f)(z') \coloneqq \int_{\Gamma \backslash \mathbb{H}} \langle f(z), \vartheta_{G,k}(z, -\overline{z'}) \rangle y^k \frac{\mathrm{d}x \mathrm{d}y}{y^2}.$$

Recall that $\vartheta_{G,k} \coloneqq \partial_h \theta^{(2)}_G$. Note that for a lattice L we have

$$\theta^{(2)}_L \left(\left(\begin{smallmatrix} z & 0 \\ 0 & z' \end{smallmatrix} \right) \right) = \theta^{(1)}_L(z) \otimes \theta^{(1)}_L(z').$$

Because of Proposition 1.6.4, for $h > 0$ also

$$(\partial_h \theta_L^{(2)})(z, z') = C \cdot \sum_{i=1}^{r} \theta_{L,P_i}^{(1)}(z) \otimes \theta_{L,\overline{P_i}}^{(1)}(z'),$$

where $(P_1, \ldots, P_r)$ is an orthonormal basis of H_m^h and C is some non-zero constant. We therefore find that

$$\Phi_{D,m,k}(f) = \mu(G)^{-1} |\operatorname{O}(D)|^{-1} C \sum_{L \in G} \frac{1}{\# \operatorname{Aut}(L)} \sum_{\sigma \in \operatorname{Iso}(D, L'/L)} \sum_{i=1}^{r} (f, \sigma^* \theta_{L,P_i}) \cdot \sigma^* \theta_{L,P_i}. \tag{4.3.1}$$

If D' is a discriminant form isomorphic to D, then for a $\tau \in \operatorname{Iso}(D', D)$ it follows from $\langle \tau^* v, \tau^* w \rangle = \langle v, w \rangle$ for all $v, w \in \mathbb{C}[D]$ that

$$\Phi_{D',m,k} \circ \tau^* = \tau^* \circ \Phi_{D,m,k}. \tag{4.3.2}$$

The following lemma will be useful.

Lemma 4.3.1. *Let V be a $\mathbb{C}$-vector space with scalar product $(\cdot, \cdot)$ which is linear in the first variable and $(v_i)_{i=1}^n \subset V$ an arbitrary finite family. Then $f : V \to V$ defined by*

$$f(v) := \sum_{i=1}^{n} (v, v_i) \cdot v_i$$

is surjective onto $\operatorname{span}(v_i)_{i=1}^n$ and is self-adjoint. In particular f is diagonalizable and

$$V = \operatorname{im}(f) \oplus \ker(f).$$

Proof. Let $v, w \in V$. Then

$$\begin{aligned} (f(v), w) &= \sum_{i=1}^{n} (v, v_i) \cdot (v_i, w) \\ &= (v, \sum_{i=1}^{n} (w, v_i) \cdot v_i) \\ &= (v, f(w)). \end{aligned}$$

Let $A = (a_{ij})$ with $a_{ij} = (v_i, v_j)$ be the Gram matrix of $(v_i)_{i=1}^n$ and define $g : \mathbb{C}^n \twoheadrightarrow \operatorname{span}(v_i)_{i=1}^n$ by

$$x = (x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} x_i \cdot v_i.$$

Then

$$x \cdot A = ((g(x), v_1), \ldots, (g(x), v_n))$$

and $g(x \cdot A) = f(g(x))$. From the first assertion we see that $\ker(g) = \ker(A)$ and since A is self-adjoint, $\mathbb{C}^n = \operatorname{im}(A) \oplus \ker(A) = \operatorname{im}(A) \oplus \ker(g)$. This implies that

$$\operatorname{im}(f \circ g) = \operatorname{im}(g(\cdot A)) = \operatorname{im}(g) = \operatorname{span}(v_i)_{i=1}^n.$$

□

Applying Lemma 4.3.1 to Φ yields

Proposition 4.3.2. *Let D, m and h be as above. If $h = 0$, assume $m > 4$. The linear map Φ is self-adjoint and surjective onto $\Theta_{m,k}(D)_0$. In particular we have*

$$\mathrm{S}_k(D) = \Theta_{m,k}(D)_0 \oplus \ker(\Phi)$$

and Φ is diagonalizable.

Proof. We begin with the case $h = 0$. By the Siegel–Weil formula we know that for $m > 4$

$$\mu(G)^{-1}|\,\mathrm{O}(D)|^{-1} \cdot \sum_{L \in G} \frac{1}{\#\operatorname{Aut}(L)} \sum_{\sigma \in \operatorname{Iso}(D, L'/L)} \sigma^* \theta_L = E_k.$$

Furthermore, the forms $\sigma^*\theta_L - E_k \in \mathrm{S}_k(D)$ span $\Theta_{m,k}(D)_0$. Since any $f \in \mathrm{S}_k(D)$ is orthogonal on E_k, we thus get

$$\begin{aligned}
&\Phi_{D,2k,k}(f) \\
&\quad = \mu(G)^{-1}|\,\mathrm{O}(D)|^{-1} \cdot \sum_{L \in G} \frac{1}{\#\operatorname{Aut}(L)} \sum_{\sigma \in \operatorname{Iso}(D, L'/L)} (f, \sigma^*\theta_L) \cdot \sigma^*\theta_L \\
&\quad = \mu(G)^{-1}|\,\mathrm{O}(D)|^{-1} \cdot \sum_{L \in G} \frac{1}{\#\operatorname{Aut}(L)} \sum_{\sigma \in \operatorname{Iso}(D, L'/L)} (f, \sigma^*\theta_L) \cdot \sigma^*\theta_L - (f, E_k)E_k \\
&\quad = \mu(G)^{-1}|\,\mathrm{O}(D)|^{-1} \cdot \sum_{L \in G} \frac{1}{\#\operatorname{Aut}(L)} \sum_{\sigma \in \operatorname{Iso}(D, L'/L)} (f, \sigma^*\theta_L) \cdot (\sigma^*\theta_L - E_k) \\
&\quad = \mu(G)^{-1}|\,\mathrm{O}(D)|^{-1} \cdot \sum_{L \in G} \frac{1}{\#\operatorname{Aut}(L)} \sum_{\sigma \in \operatorname{Iso}(D, L'/L)} (f, \sigma^*\theta_L - E_k) \cdot (\sigma^*\theta_L - E_k)
\end{aligned}$$

and we can apply Lemma 4.3.1 to the family $(\sigma^*\theta_L - E_k)_{(L,\sigma)}$.
If $h > 0$, we can immediately apply Lemma 4.3.1 because of equation (4.3.1). □

It remains to determine when Φ_D has trivial kernel. To do this we will need

Lemma 4.3.3. *Let $H \subset D$ be an isotropic subgroup and let $f \in \ker(\Phi_{H^\perp/H})$. Then also $\uparrow_H(f) \in \ker(\Phi_D)$.*

Proof. If $g \in \mathrm{im}(\Phi_D)$, then it is a linear combination of theta series, i.e.

$$g = \sum_{L\in G} \sum_{\sigma\in \mathrm{Iso}(L'/L,D)} \sum_{i=1}^{r} c_{L,\sigma,i}\sigma_*\theta_{L,P_i}.$$

Then

$$\downarrow_H(g) = \sum_{L\in G} \sum_{\sigma\in \mathrm{Iso}(L'/L,D)} \sum_{i=1}^{r} c_{L,\sigma,i}\tilde{\sigma}_* \downarrow_{\sigma^{-1}H}(\theta_{L,P_i}),$$

where $\tilde{\sigma}$ is as on page 52. Let M be the over lattice of L generated by any representatives of the elements in $\sigma^{-1}H \subset L'/L$ so that $\sigma^{-1}H = M/L$. By identifying $(M'/L)/(M/L)$ with M'/M we find that $\downarrow_{\sigma^{-1}H}(\theta_{L,P_i}) = \theta_{M,P_i}$. Hence, $\downarrow_H(g) \in \Theta_{m,k}(H^\perp/H)_0 = \mathrm{im}(\Phi_{H^\perp/H})$. Therefore, we have

$$\begin{aligned} f \in \ker(\Phi_{H^\perp/H}) &\Leftrightarrow (f,g)=0 && \forall g\in \mathrm{im}(\Phi_{H^\perp/H}) \\ &\Rightarrow (f,\downarrow_H(g))=0 && \forall g\in \mathrm{im}(\Phi_D) \\ &\Leftrightarrow (\uparrow_H(f),g)=0 && \forall g\in \mathrm{im}(\Phi_D) \\ &\Leftrightarrow \uparrow_H(f)\in \ker(\Phi_D). \end{aligned}$$

□

Combining the results of the previous sections we obtain

Theorem 4.3.4. *Let $m > 6$. The map Φ is a linear combination of Hecke operators, namely*

$$\Phi_{D,m,k} = C(m,k)\frac{e(-\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{l=1}^{\infty} \frac{T(l^2)}{l^{2k-2-h}},$$

where $C(m,k) = G_m^{k-m/2}(1,1)\frac{(k-1)!}{(m/2-1)!}C(k)$ with $C(k)$ as in (4.1.3).

Proof. Let us first consider the case $h = 0$. We have

$$\begin{aligned} \Phi_{D,2k,k}(f)(z') &= \int_{\Gamma\backslash\mathbb{H}} \langle f(z), \theta_G^{(2)}\left(\left(\begin{smallmatrix} z & 0 \\ 0 & -\overline{z'}\end{smallmatrix}\right)\right)\rangle y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2} \\ &= \int_{\Gamma\backslash\mathbb{H}} \langle f(z), E_k^{(2)}\left(\left(\begin{smallmatrix} z & 0 \\ 0 & -\overline{z'}\end{smallmatrix}\right)\right)\rangle y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2} \\ &= \int_{\Gamma\backslash\mathbb{H}} \langle f(z), E_k^{(1)}(z)\rangle y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2} \otimes E_k^{(1)}(z') \\ &\quad + \frac{e(-\operatorname{sign}(D)/8)}{\sqrt{|D|}} \int_{\Gamma\backslash\mathbb{H}} \sum_{l=1}^{\infty} \langle f(z), \omega_l(z, -\overline{z'})\rangle y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2}, \end{aligned}$$

where we have used Theorems 1.7.8 and 4.2.5. The integral defines a scalar product on the finite dimensional vector space $\mathrm{S}_k(D)$ and for a fixed z' the sum is a convergent series in the space $\mathrm{S}_k(D)$. Hence, by the continuity of the scalar product we may interchange the order of integration and summation. We use the fact that $E_k^{(1)} \in \mathrm{S}_k(D)^\perp$ and Proposition 4.1.7 to obtain

$$\begin{aligned}\Phi_{D,2k,k}(f)(z') &= \frac{e(-\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{l=1}^{\infty} \int_{\Gamma\backslash\mathbb{H}} \langle f(z), \omega_l(z, -\overline{z'})\rangle y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2} \\ &= C(k)\frac{e(-\operatorname{sign}(D)/8)}{\sqrt{|D|}} \left(\sum_{l=1}^{\infty} \frac{T(l^2)}{l^{2k-2}}\right) f.\end{aligned}$$

If $h > 0$, we first apply ∂_h and use Theorem 4.2.6 to obtain

$$\begin{aligned}&\Phi_{D,m,k}(f)(z') \\ &= \int_{\Gamma\backslash\mathbb{H}} \langle f(z), (\partial_h \theta_G^{(2)})(z, -\overline{z'})\rangle y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2} \\ &= \int_{\Gamma\backslash\mathbb{H}} \langle f(z), (\partial_h E_k^{(2)})(z, -\overline{z'})\rangle y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2} \\ &= G_m^h(1,1)\frac{(k-1)!}{(m/2-1)!} \frac{e(-\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{l=1}^{\infty} \int_{\Gamma\backslash\mathbb{H}} \langle f(z), l^h \omega_l(z, -\overline{z'})\rangle y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2} \\ &= G_m^h(1,1)\frac{(k-1)!}{(m/2-1)!} C(k)\frac{e(-\operatorname{sign}(D)/8)}{\sqrt{|D|}} \left(\sum_{l=1}^{\infty} \frac{T(l^2)}{l^{2k-2-h}}\right) f.\end{aligned}$$

□

We remark that in [67] the formula

$$\int_{\Gamma\backslash\mathbb{H}} \langle f, E_k^{(2)}\left(\left(\begin{smallmatrix} z & 0 \\ 0 & -\overline{z'}\end{smallmatrix}\right), \overline{s}\right)\rangle y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2} = \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} C(k,s) \sum_{d\in\mathbb{Z}_{>0}} d^{-k-s} T(d^2) f$$

was derived, where

$$E_k^{(2)}(Z,s) = \sum_{M\in\Gamma_\infty^{(2)}\backslash\Gamma^{(2)}} \det(\operatorname{Im}(Z))^s|_k[M]\rho_D^{(2)}(M^{-1})e^0\otimes e^0$$

and $C(k,s)$ only depends on the weight k and the variable s. However, a factor got lost: In formula (4.8) in [67] a factor of d^{k-2} must be added. Then instead of d^{-k-s} one gets $d^{-2k+2-s}$. Furthermore, the factor $e(\operatorname{sign}(D)/8)$ should be $e(-\operatorname{sign}(D)/8)$, as it was pulled out of the antilinear part of the scalar product. Then Stein's formula coincides with ours.

We further study when $\Phi_D(f)$ vanishes.

Lemma 4.3.5. *Let $m > 6$. If $\Phi_D(f) = 0$, then*

$$\prod_{p|N} \left(\sum_{r=0}^{\infty} \frac{T(p^{2r})}{p^{2rk-2r-hr}} \right) f = 0.$$

Proof. By the previous theorem and Theorem 4.1.2 we can write

$$\begin{aligned} \Phi_{D,m,k}(f) &= C(m,k) \frac{e(-\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{l=1}^{\infty} \frac{T(l^2)}{l^{2k-2-h}} f \\ &= C(m,k) \frac{e(-\operatorname{sign}(D)/8)}{\sqrt{|D|}} \left(\sum_{\substack{l=1 \\ (l,N)=1}}^{\infty} \frac{T(l^2)}{l^{2k-2-h}} \right) \prod_{p|N} \left(\sum_{r=0}^{\infty} \frac{T(p^{2r})}{p^{2rk-2r-hr}} \right) f. \end{aligned}$$

For the primes coprime to N we have seen in Proposition 4.1.3 that $\mathrm{S}_k(D)$ has a basis consisting of simultaneous Hecke eigenforms and for such an eigenform f and $\mathrm{Re}(s) > k$ we have

$$\sum_{\substack{l=1 \\ (l,N)=1}}^{\infty} \frac{T(l^2)}{l^s} f = L(f,s) \cdot f$$

with

$$L(f,s) = \prod_{p \nmid N} \frac{(1-\chi_D(p)p^{k-2-s})(1+\chi_D(p)p^{k-1-s})}{1-(\lambda(p^2)+\chi_D(p)(1-p)p^{k-2})p^{-s}+p^{2k-2-2s}}.$$

Since $L(f, 2k-2-h) \neq 0$ for $m > 4$, the operator

$$\sum_{\substack{l=1 \\ (l,N)=1}}^{\infty} \frac{T(l^2)}{l^{2k-2-h}}$$

is bijective. This proves the lemma. □

As explained in the introduction, we now show that if the conditions of the main theorem are satisfied, then for any lattice $L \in G$ there exists a sublattice $M \subset L$ with certain properties that will be useful in the proof of the main theorem.

Lemma 4.3.6. *Let D be a discriminant form of even signature* $\operatorname{sign}(D)$ *and m a positive integer such that $m = \operatorname{sign}(D) \bmod 8$ and $m >$ p-rank(D) for all primes p. Then the genus $II_{m,0}(D)$ is non-empty. Suppose for any L of genus $II_{m,0}(D)$ the $\mathbb{Z}_p$-lattice $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ splits a hyperbolic plane over $\mathbb{Z}_p$ for all primes p. Then for any L of genus $II_{m,0}(D)$, there exists a sublattice $M \subset L$ such that*

$$M'/M \cong D \oplus \langle \gamma, \mu \rangle,$$

where $n\gamma = n\mu = 0$, $\mathrm{q}(\gamma) = \mathrm{q}(\mu) = 0 \bmod 1$ and $(\gamma, \mu) = 1/n \bmod 1$.

Proof. By assumption we can write

$$L_p = \tilde{L_p} \oplus U_p,$$

where U_p is the lattice $\alpha_1\mathbb{Z}_p + \alpha_2\mathbb{Z}_p$ with Gram matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We define

$$M_p := \begin{cases} \tilde{L_p} \oplus U_p(p) & \text{if } p \mid N \\ L_p & \text{if } p \nmid N \end{cases}$$

and $M := \bigcap_{p<\infty}(M_p \cap (L \otimes_{\mathbb{Z}} \mathbb{Q}))$, where $U_p(p)$ is the lattice $p\alpha_1\mathbb{Z}_p + \alpha_2\mathbb{Z}_p$. By [41, Satz 21.5] we have $M \otimes_{\mathbb{Z}} \mathbb{Z}_p = M_p$ for all p and for $p \mid N$ we have

$$M_p'/M_p = \tilde{L_p}'/\tilde{L_p} \oplus U_p(p)'/U_p(p) \cong L_p'/L_p \oplus \langle \gamma_p, \mu_p \rangle$$

with $p\gamma_p = p\mu_p = 0$, $\mathrm{q}(\gamma_p) = \mathrm{q}(\mu_p) = 0 \bmod 1$ and $(\gamma_p, \mu_p) = 1/p \bmod 1$. This proves the result. □

We remark that a p-adic lattice L_p of rank m splits a hyperbolic plane over $\mathbb{Z}_p$ if and only if p-rank$(D) < m-2$ or p-rank$(D) = m-2$ and $\prod_q \epsilon_q = \left(\frac{-a}{p}\right)$, where the p-adic component of D is equal to

$$\bigoplus_q q^{\epsilon_q n_q}$$

and $|D| = p^\alpha a$ with $(a,p) = 1$. This is a property of the genus of L, rather than of L itself.

Finally, we can prove the main theorem.

Theorem 4.3.7. *Let D be a discriminant form of even signature* $\mathrm{sign}(D)$ *and m a positive integer such that $m = \mathrm{sign}(D) \bmod 8$, $m >$ p-rank(D) for all primes p and $m > 6$. Then there are positive-definite even lattices L such that $L'/L \cong D$, i.e. the genus $II_{m,0}(D)$ is non-empty. Suppose for any L of genus $II_{m,0}(D)$ the $\mathbb{Z}_p$-lattice $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ splits a hyperbolic plane over $\mathbb{Z}_p$ for all primes p. Then*

$$\mathrm{S}_k(D) \subset \Theta_{m,k}(D)$$

for all $k \geq m/2$.

Proof. By Proposition 4.3.2 it suffices to show that Φ_D is injective, so assume that $\Phi_D(f) = 0$ for some $f \in \mathrm{S}_k(D)$. Let $n = \prod_{p|N} p$ be the radical of N. The genus $II_{m,0}(D)$ is non-empty (see [55, Corollary 1.10.2]). We choose some $L \in II_{m,0}(D)$. By Lemma 4.3.6 there exists a lattice $M \subset L$ such that

$$\widetilde{D} := M'/M \cong D \oplus \langle \gamma, \mu \rangle,$$

where $n\gamma = n\mu = 0$, $\mathrm{q}(\gamma) = \mathrm{q}(\mu) = 0 \bmod 1$ and $(\gamma, \mu) = 1/n \bmod 1$. We set $H := \langle\gamma\rangle$ so that $D \cong H^\perp/H$. Because of (4.3.2) we can assume that $D = H^\perp/H$. According to Lemma 4.3.3 also $\Phi_{\widetilde{D}}(\uparrow_H (f)) = 0$. Then

$$\prod_{p|N}\left(\sum_{r=0}^{\infty}\frac{T(p^{2r})}{p^{2rk-2r-hr}}\right)\uparrow_H (f) = 0$$

by Lemma 4.3.5. For any $\beta \in D$ clearly $\gamma+\beta$ and $\mu+\beta$ are not in $\widetilde{D}^p$ for $p \mid N$. If $2 \mid N$, then $(n/2)\gamma, (n/2)\mu \in D_2$ and

$$\begin{aligned}2\,\mathrm{q}((n/2)\mu) + ((n/2)\mu, \gamma+\beta) &= 2\,\mathrm{q}((n/2)\gamma) + ((n/2)\gamma, \mu+\beta)\\ &= (n/2)(\mu,\gamma) = 1/2 \mod 1,\end{aligned}$$

so that $\gamma+\beta \notin \widetilde{D}^{2*}$ and $\mu+\beta \notin \widetilde{D}^{2*}$. Since $n\cdot(\gamma+\beta) = n\beta = n\cdot(\mu+\beta)$ and $\mathrm{q}(\gamma+\beta) = \mathrm{q}(\beta) = \mathrm{q}(\mu+\beta)$, we can apply Corollary 4.1.5 with set of primes $P = \{p \text{ prime} \mid p \mid N\}$. Then the element $v := v_{\gamma+\beta,\mu+\beta,P}$ from Corollary 4.1.5 is given by

$$v = \sum_{S\subset P}(-1)^{|S|}e^{\gamma_S^\mu+\beta}.$$

Note that $(\gamma_S^\mu+\beta, \gamma) = \prod_{p\in S}\frac{1}{p} \bmod 1$, so that $\downarrow_H (v) = e^\beta$ by the definition of $\downarrow_H$ (see page 51). We obtain

$$\langle f, e^\beta\rangle = \langle\uparrow_H (f), v\rangle = \langle\prod_{p|N}\left(\sum_{r=0}^{\infty}\frac{T(p^{2r})}{p^{2rk-2r-hr}}\right)\uparrow_H (f), v\rangle = 0.$$

Because β was chosen arbitrarily, it follows that $f = 0$. □

We extend the result of the main theorem to all discriminant forms of even signature and lattices of rank at least 10 by considering the space

$$\begin{aligned}\Theta^\uparrow_{m,k}(D) := \operatorname{span}\{&\uparrow_H^D (\sigma^*\theta_{L,P}) \mid L \in I\!I_{m,0}(H^\perp/H)\\ &\text{for some isotropic subgroup } H\subset D, P\in \mathrm{H}_m^{k-m/2}, \sigma\in\operatorname{Iso}(H^\perp/H, L'/L)\}.\end{aligned}$$

Then we obtain

Corollary 4.3.8. *Let D be a discriminant form of even signature* $\operatorname{sign}(D)$. *Let $m \in \mathbb{Z}_{>0}$ with $m = \operatorname{sign}(D) \bmod 8$ and $m \geq 10$. Then*

$$\mathrm{S}_k(D) \subset \Theta^\uparrow_{m,k}(D)$$

for all $k \geq m/2$.

Proof. Let $H \subset D$ be any isotropic subgroup such that all p-ranks of $H^\perp/H$ are less than or equal to 6. Then any lattice in $II_{m,0}(H^\perp/H)$ locally splits a hyperbolic plane (cf. [55, Corollary 1.9.3]). By Theorem 4.3.7 we know that $\mathrm{S}_k(H^\perp/H) \subset \Theta_{m,k}(H^\perp/H)$. We showed in Theorem 2.4.1 that

$$\begin{aligned}\mathrm{M}_k(D) = \operatorname{span}\{&\uparrow_H (f) \mid f \in \mathrm{M}_k(H^\perp/H),\\ &H \subset D \text{ isotropic subgroup such that } p\text{-rank}(H^\perp/H) \leq 6 \text{ for all primes } p\},\end{aligned}$$

which proves the corollary. □

4.4 Waldspurger's result on scalar-valued modular forms

Waldspurger's result on the scalar-valued basis problem can be derived from Theorem 4.3.7 by showing that any newform of level N is the 0-component of a suitable vector-valued cusp form. For a positive integer N let $\mathrm{S}_k^{\mathrm{new}}(N)$ denote the space of scalar-valued newforms of level N. Following [68] we let $\Theta(m,k,N,\mathcal{D})$ denote the space generated by scalar-valued theta series of positive-definite even lattices of rank m, level N and discriminant $\mathcal{D}$ weighted with harmonic polynomials of degree $k - m/2$, i.e. the 0-components of elements in $\Theta_{m,k}(D)$, where D has level N and $|D| = \mathcal{D}$.

Corollary 4.4.1. *Let m, N be positive integers.*

(i) For $m = 0 \bmod 8$ and $m \geq 8$ we have $\mathrm{S}_k^{new}(N) \subset \Theta(m,k,N,N^2)$ (cf. [68, Theorem 1]).

(ii) For $m = 4 \bmod 8$ and $m \geq 12$ and for any prime $q \mid N$ we have $\mathrm{S}_k^{new}(N) \subset \Theta(m,k,N,N^2q^2)$ (cf. [68, Theorem 2]).

Proof. By Theorem 4.3.7 it suffices to show that any $f \in \mathrm{S}_k^{\mathrm{new}}(N)$ is the 0-component of some vector-valued cusp form for an appropriate Weil representation. For a discriminant form D of level N we define a lift $\mathcal{L}_D : \mathrm{S}_k^{\mathrm{new}}(N) \to \mathrm{S}_k(D)$ by

$$\mathcal{L}_D(f) \coloneqq \sum_{M \in \Gamma_0(N)\backslash\Gamma} f|_k[M]\rho_D(M^{-1})e^0$$

and consider the map $\Psi_D : \mathrm{S}_k^{\mathrm{new}}(N) \to \mathrm{S}_k(N)$ defined by $\Psi_D(f) = \langle \mathcal{L}_D(f), e^0\rangle$. In [63, Theorem 1.1] it was shown that for certain discriminant forms, Ψ_D is a non-zero multiple of the identity map, say $c_D \cdot \mathrm{id}$.

For part (i) let $m = 0 \bmod 8$ with $m \geq 8$ and $D = (\mathbb{Z}/N\mathbb{Z})^2$ with quadratic form given by

$$(a, b) \mapsto \frac{ab}{N} \mod 1.$$

Then $II_{m,0}(D)$ is non-empty and satisfies the conditions of Theorem 4.3.7 as well as those of [63, Theorem 1.1], so that for any $f \in \mathrm{S}_k^{\mathrm{new}}(N)$ we find $\mathcal{L}_D(f) \in \Theta_{m,k}(D)$ and hence, $f = \Psi_D(c_D^{-1} f) \in \Theta(m, k, N, N^2)$.
For part (ii) let $m = 4 \bmod 8$ with $m \geq 12$ and $q \mid N$ a prime and consider the discriminant form $D = (\mathbb{Z}/N\mathbb{Z})^2 \oplus (\mathbb{Z}/q\mathbb{Z})^2$ with quadratic form given by

$$(a, b) + (c, d) \mapsto \frac{ab}{N} + \frac{c^2 - ud^2}{q} \mod 1,$$

where u is a non-square modulo q. Then $II_{m,0}(D)$ is non-empty and again, the conditions of Theorem 4.3.7 and those of [63, Theorem 1.1] are satisfied, which implies $\mathrm{S}_k^{\mathrm{new}}(N) \subset \Theta(m, k, N, N^2 q^2)$. $\square$

Theorem 3 in [68] on cuspforms with character can probably also be proved using Theorem 4.3.7, but we have not checked all the details.

Part II

Orthogonal modular forms

Chapter 5

Orthogonal modular forms and Borcherds products

In this chapter we recall some results on orthogonal modular forms including the multiplicative Borcherds lift ([7, Theorem 13.3]). We then use the main theorem of the previous chapter to prove a converse theorem for Borcherds products.

This is based on the application in the preprint [51].

5.1 Lattices and quadratic spaces

First we recall some general results on orthogonal groups. For references see for example [40] and [56]. Let R be a principal ideal domain with quotient field K of characteristic unequal to 2. Let L be an *R-lattice*, i.e. a free R-module of finite rank n together with a non-degenerate, symmetric bilinear form $(\cdot,\cdot) : L \times L \to K$ with associated quadratic form q given by $\mathrm{q}(\lambda) = (\lambda, \lambda)/2$ for $\lambda \in L$. We can retrieve $(\cdot,\cdot)$ from q since for $\lambda, \mu \in L$ we have $(\lambda, \mu) = \mathrm{q}(\lambda + \mu) - \mathrm{q}(\lambda) - \mathrm{q}(\mu)$. When R is a field, than L is called a *quadratic space*. We say that L is *even* if $(\lambda, \lambda) \in 2R$ for all $\lambda \in L$. We have already seen $\mathbb{Z}$-lattices and they will continue to play an important role in what follows. Furthermore, we will investigate $\mathbb{Z}_p$-lattices. Let A be a principal ideal domain with $R \subset A$. We extend the bilinear form linearly to the A-lattice $L_A \coloneqq L \otimes_R A$. In particular, $V \coloneqq L_K$ is called the *ambient quadratic space of* L. For $m \in R$ we denote by $L(m) \coloneqq \sqrt{m}L$ the lattice isomorphic to L with bilinear form $m(\cdot,\cdot)$.

We define the groups

$$\begin{aligned} \mathrm{GO}(L_A) &= \{\gamma \in \mathrm{GL}(L_A) \mid (\gamma v, \gamma w) = s(\gamma)(v,w) \text{ for some } s(\gamma) \in A^\times\} \\ \mathrm{O}(L_A) &= \{\gamma \in \mathrm{GO}(L_A) \mid s(\gamma) = 1\} \\ \mathrm{SO}(L_A) &= \{\gamma \in \mathrm{O}(L_A) \mid \det(\gamma) = 1\}. \end{aligned}$$

The homomorphism $s : \mathrm{GO}(L_A) \to A^\times$ is called the *similitude factor*. Let $\lambda \in L$ with $\mathrm{q}(\lambda) \in R^\times$. Then the reflection $\sigma_\lambda \in \mathrm{O}(L)$ on the hyperplane $\lambda^\perp$ is given by

$$\sigma_\lambda(v) = v - \frac{(v,\lambda)}{\mathrm{q}(\lambda)}\lambda.$$

For quadratic spaces we have (cf. e.g. [56, Theorem 43:3])

Theorem 5.1.1 (Cartan–Dieudonné Theorem)**.** *Let K be a field of characteristic unequal to 2 and let (V, q) be a quadratic space over K of dimension m. Any element in $\mathrm{O}(V)$ is the product of at most m reflections.*

If $\gamma = \sigma_{v_1} \dots \sigma_{v_k} \in \mathrm{O}(V)$ for $v_1, \dots, v_k \in V$ with $\mathrm{q}(v_i) \neq 0$, we define the *spinor norm*

$$\mathrm{spin}(\gamma) = \mathrm{q}(v_1) \dots \mathrm{q}(v_k) \mod (K^\times)^2.$$

We have (cf. [56, 54:6])

Proposition 5.1.2. *Let K be a field of characteristic unequal to 2 and let (V, q) be a quadratic space over K. Then*

$$\mathrm{spin} : \mathrm{O}(V) \to K^\times/(K^\times)^2$$

is a well-defined group homomorphism.

We define the groups

$$\begin{aligned} \mathrm{O}(L)^+ &= \{\gamma \in \mathrm{O}(L) \mid \mathrm{spin}(\gamma) = (K^\times)^2\} \\ \mathrm{SO}(L)^+ &= \{\gamma \in \mathrm{SO}(L) \mid \mathrm{spin}(\gamma) = (K^\times)^2\}. \end{aligned}$$

Finally, we describe the *Eichler transformations* introduced by Eichler in [27].

Proposition 5.1.3. *Let R be a principal ideal domain of characteristic unequal to 2 and let (L, q) be an R-lattice with ambient quadratic space V. Let $z \in L$, $z' \in V$ be isotropic with $(z, z') = -1$ and let $\mu \in L \cap z^\perp \cap z'^\perp$. Then the Eichler transformation $E_\mu^z \in \mathrm{SO}(L)^+$ is given by*

$$E_\mu^z(v) = v - (v,z)\mu + (v,\mu)z - \mathrm{q}(\mu)(v,z)z.$$

The map $\mu \mapsto E^z_\mu$ *is an injective group homomorphism from* $L \cap z^\perp \cap z'^\perp$ *to* $\mathrm{SO}(L)^+$. *We have*

$$E^z_\mu(z) = z$$
$$E^z_\mu(z') = z' + \mu + \mathrm{q}(\mu)z.$$

If $\gamma \in \mathrm{GO}(L)$, *then*

$$\gamma E^z_\mu = E^{\gamma(z)}_{s(\gamma)^{-1}\gamma(\mu)}\gamma.$$

Proof. Let $\mu, \mu' \in L \cap z^\perp \cap z'^\perp$. One verifies that

$$E^z_{\mu+\mu'} = E^z_\mu E^z_{\mu'}$$

and so $\mu \mapsto E^z_\mu$ is a group homomorphism from $L \cap z^\perp \cap z'^\perp$ to $\mathrm{O}(L)$. Injectivity follows from

$$E^z_\mu(z') = z' + \mu + \mathrm{q}(\mu)z.$$

To show that $E^z_\mu \in \mathrm{SO}(L)^+$ first, assume that $\mathrm{q}(\mu) \neq 0$. Then

$$E^z_\mu = \sigma_\mu \sigma_{\mu - \mathrm{q}(\mu)z} \in \mathrm{SO}(V)^+ \cap \mathrm{O}(L) = \mathrm{SO}(L)^+.$$

Now assume $\mathrm{q}(\mu) = 0$. For $\mu = 0$, clearly $E^z_\mu = \mathrm{id} \in \mathrm{SO}(L)^+$, otherwise let $\mu' \in L \cap z^\perp \cap z'^\perp$ be any element with $\mathrm{q}(\mu') \neq 0$, which exists since the bilinear form on L is non-degenerate. Then

$$\mathrm{q}(\mu \pm \mu') = \pm(\mu, \mu') + \mathrm{q}(\mu').$$

Since the characteristic of R is not 2, at least one of them is anisotropic. Therefore,

$$E^z_\mu = E^z_{\mu \pm \mu'} E^z_{\mp\mu'} \in \mathrm{SO}(L)^+.$$

Finally, let $\gamma \in \mathrm{GO}(L)$. One verifies that

$$\gamma E^z_\mu = E^{\gamma(z)}_{s(\gamma)^{-1}\gamma(\mu)}\gamma.$$

□

When $R \subset \mathbb{R}$, then $L_\mathbb{R} \cong \mathbb{R}^{t_+} \times \mathbb{R}^{t_-}$ such that $(\cdot, \cdot)$ is positive-definite on $\mathbb{R}^{t_+}$ and negative-definite on $\mathbb{R}^{t_-}$ and (t_+, t_-) is called the *signature* of L. We will, in particular, be interested in lattices of signature $(n, 2)$ with $n \geq 3$. In this case we must have $s(\gamma) > 0$ for all $\gamma \in \mathrm{GO}(L)$ because γ must preserve the signature of L. In particular, if $R = \mathbb{Z}$, then $\mathrm{GO}(L) = \mathrm{O}(L)$. We have

$$\mathrm{GO}(L_\mathbb{R}) = \mathbb{R}_{>0}\,\mathrm{O}(L_\mathbb{R})$$

and we can extend spin to $\mathrm{GO}(L_{\mathbb{R}})$ by $\mathrm{spin}(a \cdot \mathrm{id}) := (\mathbb{R}^\times)^2 = \mathbb{R}_{>0}$ for $a \in \mathbb{R}_{>0}$. Furthermore, note that $\mathbb{R}^\times/(\mathbb{R}^\times)^2 \xrightarrow{\sim} \{\pm 1\}$ and we will identify the two groups. Now we also define

$$\mathrm{GO}(L_A)^+ := \{\gamma \in \mathrm{GO}(L_A) \mid \mathrm{spin}(\gamma) = 1\}$$

when $A \subset \mathbb{R}$. For a $\Gamma \subset \mathrm{GO}(L_{\mathbb{R}})$ we also denote $\Gamma_+ := \{\gamma \in \Gamma \mid \det(\gamma) > 0\}$.

Recall the notion of the dual lattice $L' = \{\gamma \in V \mid (\gamma, \lambda) \in R \text{ for all } \lambda \in L\}$ of an R-lattice L. We say that L is *unimodular* if $L' = L$. In Chapter 6 we will only consider unimodular lattices and for the rest of this section we want to study orthogonal groups of unimodular lattices.

Lemma 5.1.4. *Let L be a unimodular even $\mathbb{Z}$-lattice of signature (t_+, t_-) with $t_+, t_- \geq 2$. Let z_1, z_1' and z_2, z_2' be primitive isotropic in L with $(z_1, z_1') = (z_2, z_2') = -1$. There exists a $\gamma \in \mathrm{SO}(L)^+$ with $\gamma(z_1) = z_2$ and $\gamma(z_1') = z_2'$.*

Proof. This is true for $\mathrm{O}(L)$: Let U_1 (resp. U_2) be the hyperbolic plane spanned by z_1 and z_1' (resp. z_2 and z_2'). Then $L \cap U_1^\perp$ and $L \cap U_2^\perp$ are indefinite unimodular even lattices and hence, isometric to each other (see [55, Corollary 1.13.3]). Extending any isometry between these complements by $z_1 \mapsto z_2$ and $z_1' \mapsto z_2'$ gives an element of $\mathrm{O}(L)$ with the desired property.
To prove the result for $\mathrm{SO}(L)^+$ it suffices to show that $\mathrm{O}(L \cap U_1^\perp)$ contains elements with all combinations of determinant and spinor norm. Since $L \cap U_1^\perp$ splits a hyperbolic plane, we find primitive isotropic $w, w' \in L \cap U_1^\perp$ with $(w, w') = -1$. Then

	det	spin
id	1	1
$\sigma_{w+w'}\sigma_{w-w'}$	1	-1
$\sigma_{w-w'}$	-1	1
$\sigma_{w+w'}$	-1	-1

□

Let p be a prime. We shift our attention to lattices over $\mathbb{Z}_p$. We recall some properties of $\mathbb{Z}_p$-lattices (cf. [55]). Let L_p be a unimodular $\mathbb{Z}_p$-lattice of rank m. If p is odd, then $2 \in \mathbb{Z}_p^\times$ so that L_p must be even. Furthermore, it has an orthogonal basis. If $p = 2$, we assume that L_p is even, which implies that m is even. In either case, if m is even, $L_p \cong U^k$, where U is a hyperbolic plane over $\mathbb{Z}_p$ and $k = m/2$. Note that up to isomorphism there exists only one unimodular even $\mathbb{Z}_p$-lattice of rank m.

Lemma 5.1.5. *Let U be the hyperbolic plane, spanned by the primitive isotropic vectors z, z' with $(z, z') = -1$. Let $u \in \mathbb{Z}_p^\times$. The automorphism σ of U defined by $z \mapsto uz$ and $z' \mapsto u^{-1}z'$ has determinant 1 and spinor norm $u(\mathbb{Q}_p^\times)^2$.*

Proof. The automorphism σ is the product of the reflections at $z - uz'$ and $z - z'$, i.e. $\sigma = \sigma_{z-z'}\sigma_{z-uz'}$. These reflections have spinor norms u and 1 respectively. □

Lemma 5.1.6. *Let L_2 be an even $\mathbb{Z}_2$-lattice and let z, z' and w, w' be primitive isotropic vectors in L_2 with $(z, z') = (w, w') = -1$. Then there exists a $\sigma \in \mathrm{O}(L_2)$ which is a product of reflections of the form σ_λ with $\lambda \in L_2$ and $\mathrm{q}(\lambda) \in \mathbb{Z}_2^\times$ and Eichler transformations with the property $\sigma(z) = w$, $\sigma(z') = w'$.*

Proof. We write $w = az + bz' + \lambda$ and $w' = cz + dz' + \mu$ with $\lambda, \mu \in L_2 \cap z^\perp \cap z'^\perp$ and $a, b, c, d \in \mathbb{Z}_2$. Then $\mathrm{q}(\lambda) = ab$ and $\mathrm{q}(\mu) = cd$. We distinguish several cases:
First suppose $w = z$. Then $(w, w') = -1$ implies that $d = 1$ and $\mathrm{q}(\mu) = c$. The Eichler transformation $E_\mu^z \in \mathrm{SO}(L_2)^+$ maps z to z and z' to $z' + \mathrm{q}(\mu)z + \mu = w'$.
Now consider the case that $b \in \mathbb{Z}_2^\times$. Then $\mathrm{q}(z - w) = -(z, w) = b$. Therefore, the reflection at $z - w$ has spinor norm $b \in \mathbb{Z}_2^\times$ and maps z to w and we have reduced this case to the first one.
If $a \in \mathbb{Z}_2^\times$, then the reflection at $z - z'$ has spinor norm 1 and swaps z and z' so that we can reduce to the previous case.
Similarly, if c or d is in $\mathbb{Z}_2^\times$, we can swap w and w' to reduce to one of the previous two cases.
Finally, suppose that 2 divides all of a, b, c and d. Then 4 divides $ab = \mathrm{q}(\lambda)$ and $cd = \mathrm{q}(\mu)$ and

$$-1 = (w, w') = -ad - bd + (\lambda, \mu)$$

implies $(\lambda, \mu) + 1 \in 8\mathbb{Z}_2$. Hence, the Gram matrix of λ and μ has the form

$$\begin{pmatrix} 8\mathbb{Z}_2 & -1 + 8\mathbb{Z}_2 \\ -1 + 8\mathbb{Z}_2 & 8\mathbb{Z}_2 \end{pmatrix}.$$

with determinant $-1 \bmod 16\mathbb{Z}_2$. In particular, $\langle \lambda, \mu \rangle$ is a hyperbolic plane and therefore contains an element β with $\mathrm{q}(\beta) = 0$ and $(\beta, \lambda) = -1$. Then $E_\beta^z \in \mathrm{SO}(L_2)^+$ maps z to z and z' to $z' + \beta$. Since $(w, z' + \beta) = a + (w, \beta) = a + (\lambda, \beta) = a - 1 \in \mathbb{Z}_2^\times$ we have again reduced to a previous case. □

By the Cartan–Dieudonné Theorem any element of the orthogonal group of a quadratic space over a field of characteristic not equal to 2 of dimension m is the product of at most m reflections. It is easier to simply prove that the orthogonal group is generated by reflections and the proof translates verbatim to $\mathrm{O}(L_p)$ for odd p.

Proposition 5.1.7. *Let $p > 2$. The group $\mathrm{O}(L_p)$ is generated by reflections at $\lambda^\perp$ with $\lambda \in L_p$ and $\mathrm{q}(\lambda) \in \mathbb{Z}_p^\times$.*

Proof. Let $\lambda_1, \dots, \lambda_m$ be an orthogonal basis of L_p with $\mathrm{q}(\lambda_i) \in \mathbb{Z}_p^\times$. We use induction on m:
If $m = 1$, then $\mathrm{O}(L_p) = \{\pm 1\}$, which is generated by σ_{λ_1}.
Now let $m > 1$, assume that the result is true for lattices of rank $< m$ and let $\gamma \in \mathrm{O}(L_p)$ be arbitrary. We set $\mu = \gamma(\lambda_1)$. Then

$$\mathrm{q}(\lambda_1 - \mu) + \mathrm{q}(\lambda_1 + \mu) = 2\,\mathrm{q}(\lambda_1) + 2\,\mathrm{q}(\mu) = 4\,\mathrm{q}(\lambda_1) \in \mathbb{Z}_p^\times$$

and at least one of $\sigma_{\lambda_1 - \mu}$ and $\sigma_{\lambda_1 + \mu}$ is a well-defined reflection. One checks that $\sigma_{\lambda_1 - \mu}(\lambda_1) = \mu$ and $\sigma_{\lambda_1 + \mu}\sigma_{\lambda_1}(\lambda_1) = \mu$ if it is well-defined. In either case we have found a $\sigma \in \mathrm{O}(L_p)$ which is a product of reflections such that $\sigma\gamma$ fixes λ_1. By the inductions hypothesis $\sigma\gamma$ is a product of reflections as well. □

Proposition 5.1.7 is in general not true for $p = 2$, but we have

Proposition 5.1.8. *The group $\mathrm{O}(L_2)$ is generated by reflections at $\lambda^\perp$ with $\lambda \in L_2$ and $\mathrm{q}(\lambda) \in \mathbb{Z}_2^\times$ and Eichler transformations.*

Proof. We can write

$$L_2 = \langle z, z' \rangle \oplus \langle z, z' \rangle^\perp$$

for primitive isotropic z, z' with $(z, z') = -1$. Let $\gamma \in \mathrm{O}(L_2)$ be arbitrary and set $w = \gamma(z), w' = \gamma(z')$. By Lemma 5.1.6 there exists a $\sigma \in \mathrm{O}(L_p)$ which is a product of reflections at elements in L_2 and Eichler transformations such that $\sigma(w) = z$ and $\sigma(w') = z'$. Therefore, $\sigma\gamma$ fixes $\langle z, z' \rangle$ and can be considered as an element of $\mathrm{O}(\langle z, z' \rangle^\perp)$. We use induction on $k = m/2$:
If $k = 1$ then $\langle z, z' \rangle^\perp$ is trivial and $\gamma = \sigma^{-1}$.
If $k > 1$ and the proposition is true for lattices of rank $m - 2$, then $\sigma\gamma$ is a product of reflections at elements in L_2 and Eichler transformations and hence the same is true for γ. □

From the previous two propositions immediately follows

Corollary 5.1.9. *Let L_p be a p-adic even lattice of even rank m. For any $\gamma \in \mathrm{O}(L_p)$ we have $\mathrm{spin}(\gamma) \in \mathbb{Z}_p^\times (\mathbb{Q}_p^\times)^2$.*

Now let m be even and fix a basis $(z_1, \dots, z_k, z_k', \dots, z_1')$ of L_p with

$$\mathrm{q}(z_i) = \mathrm{q}(z_i') = 0, \quad (z_i, z_i') = -1, \quad (z_i, z_j) = (z_i, z_j') = 0$$

for $i, j = 1, \dots, k$ and $i \neq j$, i.e. $\langle z_i, z_i' \rangle$ is a hyperbolic plane for each $i = 1, \dots, k$.

Let $l \in \mathbb{Z}_{>0}$ and let $0 \leq l_1 \leq \dots \leq l_k \leq l$ be integers. We define the lattice $L_p^{l_1,\dots,l_k;l}$ as the sublattice of L_p generated by

$$p^{l_1} z_1, \dots, p^{l_k} z_k, p^{l-l_k} z_k', \dots, p^{l-l_1} z_1'.$$

Proposition 5.1.10. *Let $l \in \mathbb{Z}_{>0}$ and let $M_p \subset L_p$ be a sublattice that is isomorphic to $L_p(p^l)$. Then there exist unique integers $0 \leq l_1 \leq \dots \leq l_k \leq l/2$ such that*

$$M_p \in \mathrm{O}(L_p)^+ L_p^{l_1,\dots,l_k;l}.$$

Proof. We prove the existence by induction on k.
If $k = 1$ then M_p is generated by two isotropic elements $w, w' \in L_p$ with $(w, w') = -p^l$. The only isotropic vectors in L_p are multiples of z_1 or z_1'. It follows that

$$M_p = L_p^{l_1;l}$$

for some integer l_1 with $0 \leq l_1 \leq l$. The reflection $\sigma_{z_1 - z_1'} \in \mathrm{O}(L)^+$, which swaps z_1 and z_2 has determinant -1 and spinor norm 1. Applying it to $L_p^{l_1;l}$ if necessary we can assume that $l_1 \leq l/2$.
Now let $k > 1$ and assume that the statement is true for $k-1$. We choose a primitive isotropic $w \in L_p$ with minimal l_1, where p^{l_1} is the order of

$$w + M_p \in L_p/M_p.$$

Now let $\mu \in M_p$ be any isotropic element. Then $\mu = p^{l_1'} \mu'$ for some $l_1' \geq l_1$ and $\mu' \in L_p$. Hence, for any $\lambda \in L_p$ we have

$$(p^{l-l_1}\lambda, \mu) = p^{l+l_1'-l_1}(\lambda, \mu') \in p^l \mathbb{Z}_p. \tag{5.1.1}$$

Since M_p has a basis consisting of isotropic elements, equation (5.1.1) also holds for anisotropic $\mu \in M_p$ and so for all $\mu \in M_p$. This implies $p^{l-l_1}\lambda \in M_p$ since $M_p(p^{-l})$ is unimodular. Now let $w' \in L_p$ be primitive isotropic with $(w, w') = -1$. Then $\langle p^{l_1} w, p^{l-l_1} w' \rangle$ lies in M_p and defines a hyperbolic plane rescaled by p^l. We have

$$L_p = \langle w, w' \rangle \oplus \langle w, w' \rangle^\perp$$

and since $M_p \cong L_p(p^l)$, this implies

$$M_p = \langle p^{l_1} w, p^{l-l_1} w' \rangle \oplus \widetilde{M_p},$$

where $\widetilde{M_p}$ is isomorphic to $\widetilde{L_p}(p^l)$ and $\widetilde{L_p} = \langle z_1, z_1' \rangle^\perp \cong \langle w, w' \rangle^\perp$. By the induction hypothesis there exist integers $0 \leq l_2 \leq \dots \leq l_k \leq l/2$ such that

$$\widetilde{M_p} = \gamma \left(\widetilde{L_p}^{l_2,\dots,l_k;l} \right)$$

for some $\gamma \in \mathrm{O}(\widetilde{L_p})^+$. We extend γ to an element in $\mathrm{O}(L_p)$ by

$$\begin{aligned} z_1 &\mapsto w \\ z_1' &\mapsto w'. \end{aligned}$$

Then

$$M_p = \gamma(L_p^{l_1,\dots,l_k;l})$$

and by Proposition 5.1.9 $u := \mathrm{spin}(\gamma) \in \mathbb{Z}_p^\times(\mathbb{Q}_p^\times)^2$. Let σ be the element from Lemma 5.1.5 with $z = z_1$, $z' = z_1'$. Then

$$M_p = \gamma\sigma(L_p^{l_1,\dots,l_k})$$

and $\gamma\sigma \in \mathrm{O}(L_p)^+$. Note that $l_1 \leq l_2$, since w was chosen with minimal order.
To show uniqueness assume that there are integers $0 \leq l_1 \leq \ldots \leq l_k \leq l/2$ and $0 \leq l_1' \leq \ldots \leq l_k' \leq l/2$ such that there exists some $\gamma \in \mathrm{O}(L_p)^+$ with

$$L_p^{l_1,\dots,l_k;l} = \gamma(L_p^{l_1',\dots,l_k';l}).$$

Then γ defines an isomorphism of the finite groups $L_p/L_p^{l_1,\dots,l_k;l}$ and $L_p/L_p^{l_1',\dots,l_k';l}$. We have

$$\begin{aligned} L_p/L_p^{l_1,\dots,l_k;l} &\cong \bigoplus_{i=1}^k (\mathbb{Z}/p^{l_i}\mathbb{Z} \oplus \mathbb{Z}/p^{l-l_i}\mathbb{Z}) \\ L_p/L_p^{l_1',\dots,l_k';l} &\cong \bigoplus_{i=1}^k (\mathbb{Z}/p^{l_i'}\mathbb{Z} \oplus \mathbb{Z}/p^{l-l_i'}\mathbb{Z}) \end{aligned}$$

and these groups are isomorphic if and only if $(l_1, \dots, l_k) = (l_1', \dots, l_k')$. □

Finally, we will need the following lemma for the case when $l_k < l/2$.

Lemma 5.1.11. *Let $l \in \mathbb{Z}_{>0}$ and let $0 \leq l_1 \leq \ldots \leq l_k < l/2$ be integers. Let $M_p = L_p^{l_1,\dots,l_k;l}$. If $\gamma \in \mathrm{O}(L_p)$ with $\gamma(M_p) = M_p$ then* $\det(\gamma) = 1$.

Proof. Let $\gamma(z_i) = a_i z_1 + \lambda_i$, $\gamma(z_i') = a_i' z_1 + \lambda_i'$ with $\lambda_i, \lambda_i' \in z_1'^{\perp}$ for $i = 2, \dots, k$ and $\gamma(z_1) = \sum_{i=1}^k (b_i z_i + b_i' z_i')$. Since $\gamma(M_p) = M_p$, we must have

$$p^{l_i - l_1} \mid b_i \quad \text{and} \quad p^{l - l_i - l_1} \mid b_i'.$$

We can assume that $b_1 = 1$:
Since $l_1 + l_i \leq 2l_i < l$, b_i' is divisible by p for all i. Hence, there exists some i such

that $b_i \in \mathbb{Z}_p^\times$, which implies $l_i = l_1$. Then $\sigma \in \mathrm{SO}(L_p)$ given by

$$\begin{aligned} z_1 &\mapsto z_i \\ z_1' &\mapsto z_i' \\ z_i &\mapsto z_1 \\ z_i' &\mapsto z_1' \end{aligned}$$

satisfies $\sigma(M_p) = M_p$. Replacing γ with $\sigma\gamma$ we can therefore assume that $b_1 \in \mathbb{Z}_p^\times$. Multiplying the element from Lemma 5.1.5 with $z = z_1$, $z' = z_1'$ and $u = b_1^{-1}$ to γ from the left, we can assume $b_1 = 1$.

We can assume that $\gamma(z_1) = z_1$:

Let $\mu = \sum_{i=2}^k (b_i z_i + b_i' z_i')$. The Eichler transformation $E_\mu^{z_1'} \in \mathrm{SO}(L_p)^+$ maps z_1' to z_1' and z_1 to $z_1 + \mathrm{q}(\mu) z_1' + \mu$ and so

$$E_{-\mu}^{z_1'}\gamma(z_1) = z_1 + (b_1' - \mathrm{q}(\mu))z_1'$$

since $(E_\mu^{z_1'})^{-1} = E_{-\mu}^{z_1'}$. In fact, since $E_{-\mu}^{z_1'}\gamma(z_1)$ is isotropic, we have $b_1' - \mathrm{q}(\mu) = 0$. Furthermore,

$$\begin{aligned} E_{-\mu}^{z_1'}(p^{l_1} z_1) &= p^{l_1} z_1 + p^{l_1}\,\mathrm{q}(\mu) z_1' - p^{l_1}\mu \\ &= p^{l_1} z_1 + p^{l_1}\,\mathrm{q}(\mu) z_1' - \sum_{i=2}^k (p^{l_1} b_i z_i + p^{l_1} b_i' z_i') \in M_p \\ E_{-\mu}^{z_1'}(p^{l-l_1} z_1') &= p^{l-l_1} z_1' \in M_p \end{aligned}$$

because of the conditions on b_i and b_i' stated above (Note that $p^{l-2l_1} \mid \mathrm{q}(\mu)$). For $i > 1$ we get

$$\begin{aligned} E_{-\mu}^{z_1'}(p^{l_i} z_i) &= p^{l_i} z_i + p^{l_i} b_i' z_1' \in M_p \\ E_{-\mu}^{z_1'}(p^{l-l_i} z_i') &= p^{l-l_i} z_i' + p^{l-l_i} b_i z_1' \in M_p, \end{aligned}$$

where we again used the conditions on b_i and b_i'. Replacing γ with $E_{-\mu}^{z_1'}\gamma$ we can therefore assume that $\gamma(z_1) = z_1$.

We can assume that $a_i = a_i' = 0$:

The Eichler transformation $E_{-a_i z_1}^{z_i'} \in \mathrm{SO}(L_p)^+$ maps z_i' to z_i' and z_i to $z_i - a_i z_1$ and so

$$\gamma E_{-a_i z_1}^{z_i'}(z_i) = \lambda_i$$

with $\lambda_i \in z_1'^{\perp}$. Furthermore,

$$\begin{aligned} E^{z_i'}_{-a_i z_1}(p^{l_1} z_1) &= p^{l_1} z_1 \in M_p \\ E^{z_i'}_{-a_i z_1}(p^{l-l_1} z_1') &= p^{l-l_1} z_1' + p^{l-l_1} a_i z_i' \in M_p \\ E^{z_i'}_{-a_i z_1}(p^{l_i} z_i) &= p^{l_i} z_i - p^{l_i} a_i z_1 \in M_p \\ E^{z_i'}_{-a_i z_1}(p^{l-l_i} z_i') &= p^{l-l_i} z_i' \in M_p \end{aligned}$$

and

$$\begin{aligned} E^{z_i'}_{-a_i z_1}(p^{l_j} z_j) &= p^{l_j} z_j \in M_p \\ E^{z_i'}_{-a_i z_1}(p^{l-l_j} z_j') &= p^{l-l_j} z_j' \in M_p \end{aligned}$$

for $j \notin \{1, i\}$. Therefore, we can assume that $a_i = 0$. Employing $E^{z_i}_{-a_i' z_1}$ we can also assume $a_i' = 0$.

We have showed that we can assume $\gamma(z_1) = z_1$. Since $(\gamma(z_1), \gamma(z_1')) = -1$, we must have $\gamma(z_1') = z_1' + \mathrm{q}(\lambda) z_1 + \lambda$ for some $\lambda \in z_1^{\perp} \cap z_1'^{\perp}$. We have also showed that we can assume that for any $\mu \in z_1^{\perp} \cap z_1'^{\perp}$ we have $\gamma(\mu) \in z_1'^{\perp}$ and $(\gamma(\mu), z_1) = (\gamma(\mu), \gamma(z_1)) = (\mu, z_1) = 0$, i.e. γ preserves $z_1^{\perp} \cap z_1'^{\perp}$. But this implies

$$0 = (z_1', \gamma^{-1}(\mu)) = (\gamma(z_1'), \mu) = (z_1' + \mathrm{q}(\lambda) z_1 + \lambda, \mu) = (\lambda, \mu)$$

and so $\lambda = 0$. In summary we have showed that we can assume that $\gamma(z_1) = z_1$, $\gamma(z_1') = z_1'$ and $\gamma(z_1^{\perp} \cap z_1'^{\perp}) = z_1^{\perp} \cap z_1'^{\perp}$. By induction we can thus assume that γ is the identity and hence, has determinant 1. □

5.2 Modular forms for orthogonal groups

We will now recall some results on orthogonal modular forms. References are [11] or [15].

Let L be an even $\mathbb{Z}$-lattice of signature $(n, 2)$ with bilinear form $(\cdot, \cdot)$ and quadratic form $\mathrm{q}(\lambda) = (\lambda, \lambda)/2$ for $\lambda \in L$. We assume $n \geq 3$ and that $L \otimes \mathbb{Q}$ splits two hyperbolic planes (This is always true if $n \geq 5$).

The complex manifold

$$\mathcal{K} = \{[Z_L] \in \mathbb{P}(L_{\mathbb{C}}) \mid (Z_L, Z_L) = 0, (Z_L, \overline{Z_L}) < 0\}$$

has two connected components, which are exchanged by complex conjugation of Z_L. We choose one of them and denote it by $\mathcal{K}^+$. The action of $\mathrm{GO}(L_{\mathbb{C}})$ on $L_{\mathbb{C}}$

induces an action on $\mathcal{K}$. For a subgroup $\Gamma \subset \mathrm{GO}(L_{\mathbb{R}})$ one finds that the subgroup preserving $\mathcal{K}^+$ is $\Gamma \cap \mathrm{GO}(L_{\mathbb{R}})^+$.

We describe the boundary of $\mathcal{K}^+$ in $\mathcal{N} := \{[Z_L] \in \mathbb{P}(L_{\mathbb{C}}) \mid (Z_L, Z_L) = 0\}$ (cf. [13]).

Definition 5.2.1. The boundary of $\mathcal{K}^+$ in $\mathcal{N}$ is given by

$$\partial\mathcal{K}^+ = \{[Z_L] \in \mathcal{N} \mid (Z_L, \overline{Z_L}) = 0\}$$

and for $[X + iY] \in \partial\mathcal{K}^+$ the vectors X, Y span an isotropic subspace of $L_{\mathbb{R}}$ which is either 1- or 2-dimensional. In fact, the isotropic subspaces $F \subset L_{\mathbb{R}}$ correspond to the boundary points of $\mathcal{K}^+$ in the following way.

(i) Let $F \subset L_{\mathbb{R}}$ be an isotropic line. Then F represents a boundary point of $\mathcal{K}^+$. A boundary point of this type is called special, otherwise generic. A set consisting of one special boundary point is called a zero-dimensional boundary component.

(ii) Let $F \subset L_{\mathbb{R}}$ be a two-dimensional isotropic subspace. The set of all generic boundary points which can be represented by an element of $F \otimes_{\mathbb{R}} \mathbb{C}$ is called the one-dimensional boundary component attached to F.

A boundary component is called rational if the corresponding isotropic space F is defined over $\mathbb{Q}$. We denote the set of all rational boundary components by P and write $\mathcal{K}^{+*} := \mathcal{K}^+ \cup \bigcup_{C \in P} C$. The group $\mathrm{O}(L_{\mathbb{Q}})^+$ acts on $\mathcal{K}^{+*}$. Let $\Gamma \subset \mathrm{O}(L)^+$ be of finite index. The projective variety $X_\Gamma := \Gamma \backslash \mathcal{K}^{+*}$ is the Bailey–Borel compactification of $\Gamma \backslash \mathcal{K}^+$ and the elements of $\Gamma \backslash P$ are called the cusps of X_Γ.

We denote by $A(\mathcal{K}^+)$ the affine cone over $\mathcal{K}^+$.

Definition 5.2.2. Let $\Gamma \subset \mathrm{O}(L)^+$ be of finite index, $k \in \mathbb{Z}$ and χ a character of Γ. A meromorphic function $\psi \colon A(\mathcal{K}^+) \to \mathbb{C}$ is called a meromorphic modular form of weight k and character χ for Γ if for all $Z_L \in A(\mathcal{K}^+)$

(i) $\psi(tZ_L) = t^{-k}\psi(Z_L)$ for all $t \in \mathbb{C}^\times$,

(ii) $\psi(\gamma Z_L) = \chi(\gamma)\psi(Z_L)$ for all $\gamma \in \Gamma$,

(iii) ψ is meromorphic at the boundary.

If ψ is actually holomorphic on $A(\mathcal{K}^+)$ and at the boundary, it is called a holomorphic modular form.

We remark that more generally one can define modular forms of half-integer weight. Then one would have to work with covers of $\mathcal{K}^+$ (see [11]).

Note that, since we assume $n \geq 3$, by the Koecher principle, the boundary condition is automatically fulfilled when ψ is meromorphic (resp. holomorphic) on $A(\mathcal{K}^+)$.

Let $z \in L$ be a primitive isotropic vector and $z' \in L_\mathbb{Q}$ isotropic with $(z, z') = -1$, which exists since we assume that $L_\mathbb{Q}$ splits two hyperbolic planes. Note that z represents a 0-dimensional boundary component of $\mathcal{K}^+$. We denote $K = L \cap z^\perp \cap z'^\perp$ and let

$$\mathcal{H}^\pm_{z,z'} = \{Z = X + iY \in K \otimes_\mathbb{Z} \mathbb{C} \mid (Y, Y) < 0\}.$$

For $Z \in \mathcal{H}^\pm_{z,z'}$ we define $Z_L = Z + z' + \mathrm{q}(Z)z$. Then the map $Z \mapsto [Z_L]$ is a biholomorphic map from $\mathcal{H}^\pm_{z,z'}$ to $\mathcal{K}$. We let $\mathcal{H}_{z,z'}$ be the component of $\mathcal{H}^\pm_{z,z'}$ that is mapped to $\mathcal{K}^+$. We will often simply write $\mathcal{H}$ instead of $\mathcal{H}_{z,z'}$. We remark that $\mathcal{H} = K \otimes_\mathbb{Z} \mathbb{R} + i\mathcal{C}$, where $\mathcal{C}$ is one of the connected components of

$$\{Y \in K \otimes_\mathbb{Z} \mathbb{R} \mid (Y, Y) < 0\}.$$

The action of $\mathrm{GO}(L_\mathbb{R})^+$ on $\mathcal{K}^+$ induces an action on $\mathcal{H}$. Let $\gamma \in \mathrm{GO}(L_\mathbb{R})^+$ and $Z \in \mathcal{H}$. Then $\gamma(Z_L) = j(\gamma, Z)(\gamma Z)_L$ with $j(\gamma, Z) = -(\gamma(Z_L), z) \in \mathbb{C}^\times$. The factor $j(\gamma, Z)$ satisfies the cocycle relation

$$j(\gamma_1\gamma_2, Z) = j(\gamma_1, \gamma_2 Z) j(\gamma_2, Z).$$

For a modular form ψ we define the function $\psi_z\colon \mathcal{H}_{z,z'} \to \mathbb{C}$ by $\psi_z(Z) = \psi(Z_L)$. Then

$$\psi_z(\gamma Z) = j(\gamma, Z)^k \chi(\gamma) \psi_z(Z) \tag{5.2.1}$$

for all $\gamma \in \Gamma^+$. Conversely, given a meromorphic (resp. holomorphic) $\psi_z\colon \mathcal{H}_{z,z'} \to \mathbb{C}$ which transforms as (5.2.1), we can define a modular form ψ by setting $\psi(tZ_L) = t^{-k}\psi_z(Z)$.

For any function $\psi_z\colon \mathcal{H}_{z,z'} \to \mathbb{C}$ and $\gamma \in \mathrm{GO}(L_\mathbb{R})^+$ we define

$$\psi_z|_k[\gamma](Z) := s(\gamma)^{k/2} j(\gamma, Z)^{-k} \psi(\gamma Z).$$

Note that for a scalar $\gamma = a \cdot \mathrm{id} \in \mathrm{GO}(L_\mathbb{R})^+$ we obviously have $[Z_L] = [\gamma(Z_L)]$ and therefore $\gamma Z = Z$ and $j(\gamma, Z) = -(aZ_L, z) = a = \sqrt{s(\gamma)}$. We thus have

$$\psi_z|_k[\gamma] = \psi_z$$

for all scalar γ.

Let $\mu \in K$. Then the Eichler transformation E_μ^z is in $\mathrm{SO}(L)^+$. Let $Z \in \mathcal{H}_{z,z'}$. Then

$$E_\mu^z(Z_L) = Z_L + \mu + (Z_L, \mu)z + \mathrm{q}(\mu)z = (Z+\mu)_L$$

and therefore $E_\mu^z Z = Z + \mu$ and $j(E_\mu^z, Z) = 1$. It follows that a modular form with trivial character for $\mathrm{SO}(L)^+$ has a Fourier expansion

$$\psi_z(Z) = \sum_{\lambda \in K'} a_z(\lambda) e(-(\lambda, Z))$$

in a neighborhood of z. For general congruence subgroups we have to consider appropriate sublattices $K_0 \subset K$.

We remark that the holomorphicity of ψ at the boundary is equivalent to the condition that $a_z(\lambda) \neq 0$ implies that $\lambda \in \overline{\mathcal{C}}$ in the Fourier expansion of ψ.

Let $w \in K$ be a primitive isotropic vector and choose $w' \in K_{\mathbb{Q}}$ isotropic with $(w, w') = -1$, which again exists since we assume that $L_{\mathbb{Q}}$ splits two hyperbolic planes. Then $\mathbb{Q}z + \mathbb{Q}w$ defines a one-dimensional rational boundary component. By potentially replacing w and w' with their negatives we can assume that $w + w' \in \mathcal{C}$. Then

$$\mathbb{H} \times \mathbb{H} \to \mathcal{H}, \ (\tau, \tau') \mapsto \tau w + \tau' w'$$

defines an embedding of the product of two usual complex upper half-planes $\mathbb{H}$ into $\mathcal{H}$. For $\tau = it$ we find

$$\begin{aligned} \lim_{t\to\infty} [(itw + \tau' w')_L] &= \lim_{t\to\infty} [itw + \tau' w' + z' - it\tau' z] \\ &= \lim_{t\to\infty} [w + \frac{\tau'}{it} w' + \frac{z'}{it} - \tau' z] \\ &= [w - \tau' z] \in \partial \mathcal{K}^+ \end{aligned}$$

and the one-dimensional boundary component represented by $\mathbb{Q}z + \mathbb{Q}w$ can be identified with a usual upper half-plane. The projection of a modular form to a one-dimensional boundary component is called the Siegel Φ-operator and is given by

$$\psi_z | \Phi_{w,w'} \colon \mathbb{H} \to \mathbb{C}, \ \tau \mapsto \lim_{t \to +\infty} \psi_z(itw + \tau w').$$

We set $M = K \cap w^\perp \cap w'^\perp$ and note that M is positive-definite. For more details on the cusps of X_Γ cf. [13].

Now we assume that L splits of two rescaled hyperbolic planes so that

$$L = \langle z, N_1 z' \rangle \oplus \langle w, N_2 w' \rangle \oplus M$$

for some $N_1, N_2 \in \mathbb{Z}$.

Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. We define an element $\tilde{g} \in \mathrm{O}(L_\mathbb{R})$ by

$$\begin{aligned} z &\mapsto az - cw, \\ z' &\mapsto bw' + dz', \\ w &\mapsto -bz + dw, \\ w' &\mapsto aw' + cz', \\ \lambda &\mapsto \lambda \text{ for } \lambda \in M. \end{aligned}$$

The map $g \mapsto \tilde{g}$ is continuous since it is linear as a map from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{(n+2)\times(n+2)}$. Therefore, $\tilde{g}$ is in the identity component of $\mathrm{O}(L_\mathbb{R})$, in particular $\tilde{g} \in \mathrm{SO}(L_\mathbb{R})^+$ and for $g \in \mathrm{SL}_2(\mathbb{Z})$ we have $\tilde{g} \in \mathrm{SO}(L_\mathbb{Q})^+$. Now $\tilde{g} \in \mathrm{SO}(L)^+$ if and only if $N_1/(N_1, N_2) \mid c$ and $N_2/(N_1, N_2) \mid b$, i.e. if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N_1/(N_1, N_2)) \cap \Gamma^0(N_2/(N_1, N_2))$.

Proposition 5.2.3. *Let the notation be as above and let ψ be a holomorphic modular form with trivial character for $\mathrm{SO}(L)^+$ and Fourier expansion*

$$\psi_z(Z) = \sum_{\lambda \in K'} a(\lambda) e(-(\lambda, Z)).$$

Then $\psi_z|\Phi_{w,w'}$ is a modular form of weight k for $\Gamma_0(N_1/(N_1, N_2)) \cap \Gamma^0(N_2/(N_1, N_2))$. Its Fourier expansion is given by

$$\psi_z|\Phi_{w,w'}(\tau) = \sum_{n \geq 0} a(nw/N_2) e(n\tau/N_2).$$

Proof. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N_1/(N_1, N_2)) \cap \Gamma^0(N_2/(N_1, N_2))$. Then

$$\begin{aligned} \psi_z|\Phi_{w,w'}\left(\frac{a\tau + b}{c\tau + d}\right) &= \lim_{t \to +\infty} \psi_z\left(itw + \frac{a\tau + b}{c\tau + d} w'\right) \\ &= \lim_{t \to +\infty} \psi\left(itw + \frac{a\tau + b}{c\tau + d} w' + z' - it\frac{a\tau + b}{c\tau + d} z\right) \\ &= (c\tau + d)^k \lim_{t \to +\infty} \\ &\qquad \psi((c\tau + d)itw + (a\tau + b)w' + (c\tau + d)z' - it(a\tau + b)z) \\ &= (c\tau + d)^k \lim_{t \to +\infty} \psi(\tilde{g}(itw + \tau w' + z' - it\tau z)) \\ &= (c\tau + d)^k \lim_{t \to \infty} \psi_z(itw + \tau w') \\ &= (c\tau + d)^k \psi_z|\Phi_{w,w'}(\tau). \end{aligned}$$

We consider the Fourier expansion

$$\begin{aligned} \psi_z|\Phi_{w,w'}(\tau) &= \lim_{t \to +\infty} \psi_z(itw + \tau w') \\ &= \sum_{\lambda \in K' \cap \overline{\mathcal{C}}} a(\lambda) \lim_{t \to +\infty} e(-(\lambda, itw + \tau w')). \end{aligned}$$

The dual lattice K' is $\frac{1}{N_2}\mathbb{Z}w + \mathbb{Z}w' + M'$, where M' is the dual of M. Therefore

$$\begin{aligned}\psi_z|\Phi_{w,w'}(\tau) &= \sum_{\substack{n,m\geq 0\\ \mu\in M'\\ \mathrm{q}(\mu)\leq \frac{nm}{N_2}}} a(nw/N_2 + mw' + \mu)e(n\tau/N_2)\lim_{t\to+\infty} e^{-2\pi mt}\\ &= \sum_{n\geq 0} a(nw/N_2)e(n\tau/N_2),\end{aligned}$$

where in the last step we used that the limit is zero for $m > 0$ and M is positive-definite. □

If ψ transforms with character det, it is trivial on all boundary components corresponding to achiral lattices.

Lemma 5.2.4. *Let the notation be as above and let ψ be a modular form for $\mathrm{O}(L)^+$ with character* det. *If there exists a $\gamma \in O(M)$ with $\det(\gamma) = -1$, then $\psi_z|\Phi_{w,w'} = 0$.*

Proof. We extend γ to an element in $\mathrm{O}^+(L)$ by letting it act trivially on $L \cap M^\perp$. Let $Z = itw + \tau w'$. Then $j(\gamma, Z) = -(\gamma(Z + z' - it\tau z), z) = -(Z + z' - it\tau z, z) = 1$. Therefore,

$$\begin{aligned}\psi_z|\Phi_{w,w'}(\tau) &= \lim_{t\to+\infty} \psi_z(itw + \tau w')\\ &= \det(\gamma)\lim_{t\to+\infty} \psi_z|_k[\gamma](itw + \tau w')\\ &= \det(\gamma)\lim_{t\to+\infty} \psi_z(itw + \tau w')\\ &= \det(\gamma)\psi_z|\Phi_{w,w'}(\tau).\end{aligned}$$

□

5.3 Borcherds products and a converse theorem

In [7] Borcherds constructed a lift from modular forms of weight $1-n/2$ for the Weil representation of $\mathrm{Mp}_2(\mathbb{Z})$ to meromorphic orthogonal modular forms for finite index subgroups Γ of $\mathrm{O}(L)^+$ (see [7, Theorem 13.3]). These orthogonal modular forms have product expansions at 0-dimensional cusps and are therefore called automorphic products or Borcherds products.

In this section we want to describe the Borcherds lift. We will then apply Theorem 4.3.7 on the vector-valued basis problem to show that the space of local obstructions for constructing Borcherds products generates the space of global obstructions. This extends Theorem 5.4 in [13].

Recall Definition 1.3.1 of modular forms for the Weil representation. We also want to allow meromorphicity at the cusps. For $k \in \frac{1}{2}\mathbb{Z}$ and a discriminant form D we let $\mathrm{M}^!_k(D)$ be the space of holomorphic functions $f : \mathbb{H} \to \mathbb{C}[D]$ such that $f|_k[(M,\phi)] = \rho_D((M,\phi))f$ for all $(M,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$ and f is meromorphic at the cusp ∞.

There is a natural map $\mathrm{O}(L) \to \mathrm{O}(L'/L)$ and for a $f \in \mathrm{M}^!_k(L'/L)$ we denote

$$\mathrm{O}(L,f)^+ := \{\gamma \in \mathrm{O}(L)^+ \mid \gamma^* f = f\}.$$

Now we have

Theorem 5.3.1 (Theorem 13.3, [7]). *Let L be an even lattice of signature $(n,2)$ and $f \in \mathrm{M}^!_{1-n/2}(L'/L)$ with integer Fourier coefficients $c(\gamma,m)$ for $m \le 0$. Then there is a meromorphic function $\Psi_L(\cdot;f)$ on $A(\mathcal{K}^+)$ called the multiplicative Borcherds lift with the following properties.*

1. *$\Psi_L(\cdot;f)$ is a modular form of weight $c(0,0)/2$ for the group $\mathrm{O}(L,f)^+$ with respect to some unitary character χ of $\mathrm{O}(L,f)^+$.*

2. *The only zeros or poles of Ψ_L lie on the rational quadratic divisors $\lambda^\perp$ for $\lambda \in L$ with $\mathrm{q}(\lambda) > 0$ and are zeros of order*

   $$\sum_{\substack{0<x\in\mathbb{R}\\ x\lambda\in L'}} c(x\lambda, -x^2\,\mathrm{q}(\lambda))$$

   (or poles if this number is negative).

3. *If $c(0,0) = n-2$ then the only nonzero Fourier coefficients of Ψ_L correspond to vectors of K of norm 0.*

4. *For each primitive norm 0 vector z of L and for each Weyl chamber W of K the function $\Psi_z(Z;f)$ has an infinite product expansion converging when Z is in a neighborhood of the cusp of z and $Y \in W$ which is up to a constant*

   $$e(-(Z,\rho)) \prod_{\substack{\lambda\in K'\\ (\lambda,W)>0}} \prod_{\substack{\mu\in L'/L\\ \mu|_{L\cap z^\perp}=\lambda}} (1 - e(-(\mu,z') - (\lambda,Z)))^{c(\mu,-\mathrm{q}(\lambda))},$$

   where ρ is the Weyl vector corresponding to W.

We remark that for $c(0,0)$ odd we get orthogonal modular forms of half-integer weight, which we have not defined in this text.

For the rest of this chapter let L be an even lattice of signature $(n, 2)$ with even $n \geq 3$ and assume that L splits two hyperbolic planes $II_{1,1} \oplus II_{1,1}$. Let $\Gamma \subset \mathrm{O}(L)^+$ be of finite index.

A *divisor* on X_Γ is a formal linear combination $\sum n_Y Y$ $(n_Y \in \mathbb{Z})$ of irreducible closed analytic subsets Y of codimension 1 such that the support $\bigcup_{n_Y \neq 0} Y$ is a closed analytic subset of everywhere pure codimension 1. For any vector $\lambda \in L'$ of positive norm the orthogonal complement of λ in $\mathcal{K}^+$ defines a divisor $\lambda^\perp$ on $\mathcal{K}^+$. Let $\beta \in L'/L$ and $l \in \mathbb{Z} + \mathrm{q}(\beta)$ with $l > 0$. Then

$$H(\beta, l) = \sum_{\substack{\lambda \in \beta + L \\ \mathrm{q}(\lambda) = l}} \lambda^\perp$$

is a Γ-invariant divisor on $\mathcal{K}^+$. It determines a divisor on $\Gamma^+ \backslash \mathcal{K}^+$ and by taking the closure also on X_Γ. Following Borcherds we call this divisor *Heegner divisor* of discriminant (β, l). Note that $H(\beta, l) = H(-\beta, l)$. It defines an element of the *divisor class group* $\mathrm{Cl}(X_\Gamma)$, which is the group of divisors modulo linear equivalence. Two divisors are called linearly equivalent if their difference is principal, i.e. the divisor of a meromorphic function on X_Γ. The *Picard group* $\mathrm{Pic}(X_\Gamma)$ of X_Γ is the group of isomorphism classes of line bundles on X_Γ with the inner product given by the tensor product. We have $\mathrm{Pic}(X_\Gamma) \hookrightarrow \mathrm{Cl}(X_\Gamma)$, so that we can write any line bundle as a divisor.

Let $F \subset L \otimes_{\mathbb{Z}} \mathbb{Q}$ be any 2-dimensional isotropic subspace and let $\tilde{F} \subset L \otimes_{\mathbb{Z}} \mathbb{Q}$ be a complementary subspace such that $F + \tilde{F}$ is the sum of two hyperbolic planes. We assume that $(F + \tilde{F}) \cap L = II_{1,1} \oplus II_{1,1}$ and set $M = L \cap F^\perp \cap \tilde{F}^\perp$. Then M is positive-definite of rank $n - 2$ and $D := L'/L \cong M'/M$. There exists a natural isomorphism $\pi : D \to M'/M$ induced from the projection from L' to M'. Let s be a generic boundary point on the 1-dimensional boundary component corresponding to F. We define the local Picard group of X_Γ in s by

$$\mathrm{Pic}(X_\Gamma, s) = \varinjlim \mathrm{Pic}(U_{\mathrm{reg}}),$$

where U ranges through all open neighborhoods of s and $U_{\mathrm{reg}} = U \cap (\Gamma \backslash \mathcal{K}^+)$. For $\Gamma_\infty = P \cap \Gamma$, where $P \subset \mathrm{O}(L \otimes_{\mathbb{Z}} \mathbb{R})$ is the parabolic subgroup stabilizing $F \otimes \mathbb{R}$, we can consider the pullback of divisors on $\Gamma \backslash \mathcal{K}^+$ to divisors on $\Gamma_\infty \backslash U_\epsilon$ for certain neighborhoods U_ϵ (cf. [13, Section 4]). The pullback of $H(\beta, l)$ is denoted by $H_F(\beta, l)$ and will be called a *local Heegner divisor*. One can show that

$$H_F(\beta, l) = \sum_{\substack{\lambda \in (\beta + L) \cap F^\perp \\ \mathrm{q}(\lambda) = l}} \lambda^\perp$$

and $H_F(\beta, l)$ defines an element of $\mathrm{Pic}(X_\Gamma, s)$ (for details see [13]). The following proposition was shown in [13].

Proposition 5.3.2 (Proposition 5.1, [13]). *A finite linear combination of divisors*

$$\frac{1}{2}\sum_{\beta\in D}\sum_{\substack{l\in\mathbb{Z}+\mathrm{q}(\beta)\\ l>0}} c(\beta,l)H_F(\beta,l)$$

(with $c(\beta,l)\in\mathbb{Z}$ and $c(\beta,l)=c(-\beta,l)$) is a torsion element of $\mathrm{Pic}(X_\Gamma, s)$ if and only if

$$\sum_{\beta\in D}\sum_{\substack{l\in\mathbb{Z}+\mathrm{q}(\beta)\\ l>0}} c(\beta,l)a(\pi(\beta),l)=0$$

for all theta series $\theta_{M,P}$ with spherical polynomial $P\in\mathrm{H}^2_{n-2}$ and Fourier coefficients $a(\beta,l)$ ($\beta\in M'/M$ and $l\in\mathbb{Z}+\mathrm{q}(\beta)$).

Let from now on $\Gamma=\ker(\mathrm{O}(L)\to\mathrm{O}(L'/L))\cap\mathrm{O}(L)^+$ be the discriminant kernel of L. By Theorem 5.3.1 the Borcherds lift of a $f\in\mathrm{M}^!_{1-n/2}(D)$ is a meromorphic modular form for the group Γ. Its divisor is a linear combination of Heegner divisors. The following characterization can be found in [13, Theorem 5.2] and goes back to Borcherds (cf. Theorem 3.1 [8]).

Theorem 5.3.3 (Borcherds). *A finite linear combination of Heegner divisors*

$$\frac{1}{2}\sum_{\beta\in D}\sum_{\substack{l\in\mathbb{Z}+\mathrm{q}(\beta)\\ l>0}} c(\beta,l)H(\beta,l)$$

(with $c(\beta,l)\in\mathbb{Z}$ and $c(\beta,l)=c(-\beta,l)$) is the divisor of a Borcherds product for the group Γ if and only if for any cusp form $f\in\mathrm{S}_{1+n/2}(D)$ with Fourier coefficients $a(\beta,l)$ the equality

$$\sum_{\beta\in D}\sum_{\substack{l\in\mathbb{Z}+\mathrm{q}(\beta)\\ l>0}} c(\beta,l)a(\beta,l)=0$$

holds.

According to Theorem 5.3.3 the space $\mathrm{S}_{1+n/2}(D)$ carries some information on the subgroup of $\mathrm{Pic}(X_\Gamma)$ generated by the divisors of Borcherds products, while the space generated by the theta series $\theta_{M,P}$ carries information on the local Picard group $\mathrm{Pic}(X_\Gamma, s)$. So when the theta series span the space of cusp forms, then we can infer that a linear combination of Heegner divisors is the divisor of a Borcherds product if and only if it is locally trivial.

Definition 5.3.4. A divisor H on X_Γ is called trivial at generic boundary points if for every one-dimensional irreducible component B of the boundary of X_Γ there exists a generic point $s \in B$ such that H is a torsion element of $\mathrm{Pic}(X_\Gamma, s)$.

The following result was suggested by Jan Bruinier and generalizes [13, Theorem 5.4] to non-unimodular lattices.

Theorem 5.3.5. *Let L be an even lattice of signature $(n, 2)$ with even $n > 8$ splitting two hyperbolic planes $II_{1,1} \oplus II_{1,1}$. Assume that the discriminant form $D = L'/L$ satisfies the conditions of Theorem 4.3.7 for $m = n - 2$. Let*

$$H = \frac{1}{2} \sum_{\beta \in D} \sum_{\substack{l \in \mathbb{Z} + \mathrm{q}(\beta) \\ l > 0}} c(\beta, l) H(\beta, l)$$

be a finite linear combination of Heegner divisors $H(\beta, l)$ (with coefficients $c(\beta, l) \in \mathbb{Z}$). Then the following statements are equivalent:

i) H is the divisor of a Borcherds product for the group Γ.

ii) H is the divisor of a meromorphic automorphic form for Γ.

iii) H is trivial at generic boundary points.

Proof. The modularity of a meromorphic modular form ψ for the group Γ immediately implies that the divisor (ψ) attached to ψ is trivial at generic boundary points. Therefore, we only need to prove (iii) implies (i). So assume that H is trivial at generic boundary points. Note that, since L splits two hyperbolic planes, the genus of L contains only one class and the natural projection $\mathrm{O}(L) \to \mathrm{O}(L'/L)$ is surjective (cf. [55, Theorem 1.14.2]). So by proposition 5.3.2

$$\sum_{\beta \in D} \sum_{\substack{l \in \mathbb{Z} + \mathrm{q}(\beta) \\ l > 0}} c(\beta, l) a(\beta, l) = 0$$

for any cusp form $f \in \Theta_{m,1+n/2}(D)$ with Fourier coefficients $a(\beta, l)$ ($\beta \in D$ and $l \in \mathbb{Z} + \mathrm{q}(\beta)$). Now according to Theorem 4.3.7 $\Theta_{m,1+n/2}(D) = \mathrm{S}_{1+n/2}(D)$ and so Theorem 5.3.3 implies that H is the divisor of a Borcherds product. $\square$

As a corollary we find for lattices L that satisfy the conditions of Theorem 5.3.5 that any meromorphic modular form for the discriminant kernel of L, whose divisor is a linear combination of Heegner divisors, is a Borcherds product. This was already proved in greater generality in [11], however, using an entirely different argument, which says nothing about the local Picard groups.

Chapter 6

Orthogonal Hecke operators

In this chapter we will investigate orthogonal Hecke operators. While Hecke operators acting on modular forms for symplectic groups have been studied intensively, orthogonal Hecke operators have only recently been investigated (see [33] and [43]). First we will study the Hecke algebra of double cosets of orthogonal similitude matrices. In particular, we will prove two elementary divisor theorems describing a canonical representative of a given double coset. Next we prove some results on the Hecke operator $\mathcal{T}(p)$. We will give an explicit decomposition of the corresponding double coset into right cosets and describe how this operator commutes with the Φ-operators.

This chapter is based on joint work with Moritz Dittmann and Nils Scheithauer (cf. [22]).

Throughout this chapter we let L be an even lattice of signature $(n, 2)$ with $n \geq 3$ and with bilinear form $(\cdot, \cdot)$ and corresponding quadratic form $\mathrm{q}(\lambda) = (\lambda, \lambda)/2$. We assume that L is unimodular. Therefore, L splits two hyperbolic planes $II_{1,1} \oplus II_{1,1}$ and $n = 2 \mod 8$. We also assume that the character χ is either trivial or equal to det. If ψ is a modular form of weight k and character χ for $\mathrm{O}(L)^+$, then ψ is also a modular form without character for $\mathrm{SO}(L)^+$. If $\chi = \det$ we extend this character to $\mathrm{GO}(L_{\mathbb{Q}})$ by $\chi(\alpha) = \mathrm{sgn}(\det(\alpha))$.

6.1 The orthogonal Hecke algebra

Let us first recall some facts about Hecke algebras as introduced by Shimura. A nice reference is [30].

Let G be a group and $\Gamma \subset G$ be a subgroup. We say that (Γ, G) is a *Hecke pair* if for all $a \in G$ the double coset $\Gamma a \Gamma$ is the union of finitely many right cosets

$$\Gamma a \Gamma = \bigcup_{i=1}^{h} \Gamma a_i.$$

Now let $\mathscr{L}(\Gamma, G)$ and $\mathscr{H}(\Gamma, G)$ be the free $\mathbb{Z}$-modules generated by the elements of $\Gamma \backslash G$ and $\Gamma \backslash G / \Gamma$ respectively. An element $a \in G$ acts on $\mathscr{L}(\Gamma, G)$ by right multiplication. We denote by

$$\mathscr{L}(\Gamma, G)^{\Gamma} := \{T \in \mathscr{L}(\Gamma, G) \mid T\gamma = T \text{ for } \gamma \in \Gamma\}$$

the submodule of Γ-invariants. Then the map given by

$$\Gamma a \Gamma \mapsto \sum_{i=1}^{h} \Gamma a_i$$

defines an isomorphism $\mathscr{H}(\Gamma, G) \xrightarrow{\sim} \mathscr{L}(\Gamma, G)^{\Gamma}$ and we will identify these two modules. Let $a, b \in G$ and

$$\Gamma a \Gamma = \bigcup_{i} \Gamma a_i, \quad \Gamma b \Gamma = \bigcup_{j} \Gamma b_j.$$

We define the product of $\Gamma a \Gamma$ and $\Gamma b \Gamma$ to be

$$(\Gamma a \Gamma) \cdot (\Gamma b \Gamma) = \sum_{i,j} \Gamma a_i b_j \in \mathscr{L}(\Gamma, G).$$

Clearly, $(\Gamma a \Gamma) \cdot (\Gamma b \Gamma) \in \mathscr{L}(\Gamma, G)^{\Gamma}$ and so this extends to a well-defined product on $\mathscr{H}(\Gamma, G)$. We get

Theorem 6.1.1. *The Hecke algebra $\mathscr{H}(\Gamma, G)$ of the pair (Γ, G) is an associative $\mathbb{Z}$-algebra with one-element $\Gamma 1 \Gamma = \Gamma$.*

An *antihomomorphism* of the pair (Γ, G) is a map

$$G \to G, \quad a \mapsto a',$$

with the properties

(i) $(a')' = a$,

(ii) $(ab)' = b'a'$,

(iii) $a \in \Gamma \Rightarrow a' \in \Gamma$.

We can extend an antihomomorphism of the pair (Γ, G) linearly to a map on $\mathscr{H}(\Gamma, G)$. If for every $a \in G$ we have $\Gamma a \Gamma = \Gamma a' \Gamma$, then the extension to $\mathscr{H}(\Gamma, G)$ is just the identity and so for $T_1, T_2 \in \mathscr{H}(\Gamma, G)$ we have

$$T_1 T_2 = (T_1 T_2)' = T_2' T_1' = T_2 T_1.$$

Thus, we have

Theorem 6.1.2. *If the Hecke pair (Γ, G) has an antihomomorphism with the property $\Gamma a \Gamma = \Gamma a' \Gamma$ for every $a \in G$ then $\mathscr{H}(\Gamma, G)$ is commutative.*

We now want to show that the pairs

$$\begin{aligned}&(\mathrm{O}(L), \mathrm{GO}(L_{\mathbb{Q}})),\\&(\mathrm{O}(L)^+, \mathrm{GO}(L_{\mathbb{Q}})^+),\\&(\mathrm{SO}(L), \mathrm{GO}(L_{\mathbb{Q}})_+),\\&(\mathrm{SO}(L)^+, \mathrm{GO}(L_{\mathbb{Q}})_+^+)\end{aligned}$$

are Hecke pairs. Let (Γ, G) be one of the four pairs above. Note that for any $\alpha \in G$, for some $N \in \mathbb{Z}$ we have $N\alpha(L) \subset L$. If

$$\Gamma \alpha \Gamma = \bigcup_{i=1}^{h} \Gamma \alpha_i$$

then also

$$\Gamma N \alpha \Gamma = \bigcup_{i=1}^{h} \Gamma N \alpha_i$$

for any $N \in \mathbb{Q}$. In order to show that $\Gamma \backslash \Gamma \alpha \Gamma$ is finite, we can therefore always assume that $\alpha(L) \subset L$. Let $m \in \mathbb{Z}_{>0}$. We denote

$$G(m) = \{\alpha \in G \mid \alpha(L) \subset L,\ s(\alpha) = m\}$$

and recall that $L(m) \coloneqq \sqrt{m}L$ is the lattice isomorphic to L with the bilinear form rescaled by m. We have

Lemma 6.1.3. *Let $m \in \mathbb{Z}_{>0}$. Then $L(m)$ is an even lattice and*

$$L(m)'/L(m) \cong \bigoplus_{p^k \| m} \left(p^k\right)^{+(n+2)}.$$

If $\alpha \in G(m)$, then $\alpha(L) \cong L(m)$ and $m\alpha^{-1} \in G(m)$.

Proof. Since L is unimodular, $(\sqrt{m}L)' = \frac{1}{\sqrt{m}}L$. So as groups

$$L(m)'/L(m) \cong (\mathbb{Z}/m\mathbb{Z})^{n+2}$$

and the level of $L(m)$ is clearly m so that we can write

$$L(m)'/L(m) \cong \bigoplus_{p^k \| m} \left(p^k\right)^{\epsilon_p(n+2)}.$$

The determinant of $L(m)$ (determinant of any Gram matrix of $L(m)$) is m^{n+2}. If we write $m = m'p^k$ with $(m', p) = 1$, then $\epsilon_p = \left(\frac{m'^{n+2}}{p}\right) = +1$ since n is even (cf. [20, Chapter 15, eq. (29)]).
If $\alpha \in G(m)$, then $\sqrt{m}\alpha^{-1}$ defines an isometry $\alpha(L) \xrightarrow{\sim} \sqrt{m}L$. Now $\alpha(L) \subset L \subset \alpha(L)'$ and since $L(m)'/L(m)$ has exponent m, also $mL \subset \alpha(L)$, hence $m\alpha^{-1}(L) \subset L$. Therefore, $m\alpha^{-1} \in G(m)$ since $s(m\alpha^{-1}) = m$. □

Whether the above pairs are Hecke pairs is essentially equivalent for the four pairs.

Lemma 6.1.4. *Let $\alpha_i \in \mathrm{GO}(L_{\mathbb{Q}})^+_+(m)$ for $i \in I$ for some index set I. Then the following are equivalent*

(i) $\mathrm{GO}(L_{\mathbb{Q}})(m) = \bigcup_{i\in I} \mathrm{O}(L)\alpha_i$ *is a disjoint union.*

(ii) $\mathrm{GO}(L_{\mathbb{Q}})^+(m) = \bigcup_{i\in I} \mathrm{O}^+(L)\alpha_i$ *is a disjoint union.*

(iii) $\mathrm{GO}(L_{\mathbb{Q}})_+(m) = \bigcup_{i\in I} \mathrm{SO}(L)\alpha_i$ *is a disjoint union.*

(iv) $\mathrm{GO}(L_{\mathbb{Q}})^+_+(m) = \bigcup_{i\in I} \mathrm{SO}^+(L)\alpha_i$ *is a disjoint union.*

Proof. Let (Γ, G) and (Γ', G') be two of the four pairs. Assume that the assertion holds for (Γ, G). We want to show that it also holds for (Γ', G').
Let $\alpha \in G'(m)$. There exists an element $\gamma \in \mathrm{O}(L)$ such that $\gamma\alpha \in G(m)$. Hence, there exist $i \in I$ and $\gamma' \in \Gamma$ such that $\gamma'\alpha_i = \gamma\alpha$ and $\gamma^{-1}\gamma'\alpha_i = \alpha$. Because $\alpha_i \in \mathrm{GO}(L_{\mathbb{Q}})^+_+(m)$, we find $\mathrm{spin}(\gamma^{-1}\gamma') = \mathrm{spin}(\alpha)$ and $\det(\gamma^{-1}\gamma') = \mathrm{sgn}(\det(\alpha))$ so that $\gamma^{-1}\gamma' \in \Gamma'$.
Now assume that $\alpha \in \Gamma'\alpha_i \cap \Gamma'\alpha_j$ for some $i, j \in I$. Then there exist $\gamma_i, \gamma_j \in \Gamma'$ such that $\gamma_i\alpha_i = \alpha = \gamma_j\alpha_j$. Therefore, $\alpha_i = \gamma_i^{-1}\gamma_j\alpha_j$ and $\mathrm{spin}(\gamma_i^{-1}\gamma_j) = \det(\gamma_i^{-1}\gamma_j) = 1$, so that $\gamma_i^{-1}\gamma_j \in \mathrm{SO}(L)^+ \subset \Gamma$ and $i = j$. □

For two lattices L, M we define

$$\{M \subset L\} := \{K \subset L \mid K \cong M\},$$

i.e. the set of sublattices of L isometric to M. We have

Proposition 6.1.5. *Let (Γ, G) be one of the pairs above and let $m \in \mathbb{Z}_{>0}$. Then The maps*

$$\begin{aligned} \phi_1 : G(m)/\Gamma &\to \{L(m) \subset L\} \\ \alpha &\mapsto \alpha(L) \end{aligned}$$

and

$$\begin{aligned} \phi_2 : \Gamma\backslash G(m) &\to \{L(m) \subset L\} \\ \alpha &\mapsto m\alpha^{-1}(L) \end{aligned}$$

are bijections.

Proof. By Lemma 6.1.3 the map

$$\begin{aligned} i : \Gamma\backslash G(m) &\to G(m)/\Gamma \\ \alpha &\mapsto m\alpha^{-1} \end{aligned}$$

is a bijection and $\phi_2 = \phi_1 \circ i$. So it suffices to prove the statement for ϕ_1. We first consider the case $(\Gamma, G) = (\mathrm{O}(L), \mathrm{GO}(L_{\mathbb{Q}}))$.
The map is well-defined by Lemma 6.1.3. Now let $M \in \{L(m) \subset L\}$. By definition there exists an isometry $\alpha : L(m) \to M$. One verifies that $\alpha(\sqrt{m}\cdot) \in G(m)$. So ϕ_1 is surjective. Now assume that for $\alpha_1, \alpha_2 \in G(m)$ we have $\alpha_1(L) = \alpha_2(L)$. Then $\alpha_2^{-1}(\alpha_1(L)) = L$, so that $\alpha_2^{-1} \circ \alpha_1 \in \mathrm{O}(L)$ and ϕ_1 is injective.
Now let (Γ, G) be arbitrary. By Lemma 6.1.4 the natural inclusions

$$\mathrm{SO}(L)^+ \backslash \mathrm{GO}(L_{\mathbb{Q}})^+_+(m) \to \Gamma\backslash G(m) \to \mathrm{O}(L)\backslash \mathrm{GO}(L_{\mathbb{Q}})(m)$$

are bijections and the statement follows from the first case. ☐

Now we have

Proposition 6.1.6. *The pairs*

$$\begin{aligned} &(\mathrm{O}(L), \mathrm{GO}(L_{\mathbb{Q}})), \\ &(\mathrm{O}(L)^+, \mathrm{GO}(L_{\mathbb{Q}})^+), \\ &(\mathrm{SO}(L), \mathrm{GO}(L_{\mathbb{Q}})_+), \\ &(\mathrm{SO}^+(L), \mathrm{GO}(L_{\mathbb{Q}})^+_+) \end{aligned}$$

are Hecke pairs.

Proof. By the previous proposition it suffices to show that for any $m \in \mathbb{Z}_{>0}$ the set $\{L(m) \subset L\}$ is finite. Note that up to isomorphism there exists only one unimodular lattice of signature $(n, 2)$ and so

$$\begin{aligned}\{L(m) \subset L\} &\xrightarrow{\frac{1}{\sqrt{m}}\cdot} \{L(m) \subset M \subset L(m)' \mid M \text{ is unimodular}\} \\ &\to \{H \subset L(m)'/L(m) \mid H \text{ self-dual, isotropic}\}\end{aligned}$$

are bijections. Since $L(m)'/L(m)$ is a finite module, the last set is finite. □

We are in particular interested in the Hecke pairs

$$(\mathrm{O}(L)^+, \mathrm{GO}(L_{\mathbb{Q}})^+) \quad \text{and} \quad (\mathrm{SO}(L)^+, \mathrm{GO}(L_{\mathbb{Q}})^+_+)$$

Let (Γ, G) be one of the two and let χ be trivial or det. Let ψ be a modular form of weight k and character χ for Γ. For $\alpha \in G$ and

$$\Gamma\alpha\Gamma = \bigcup_{i=1}^{h} \Gamma\alpha_i$$

we define

$$\psi_z|_k\Gamma\alpha\Gamma = \sum_{i=1}^{h} \chi(\alpha_i^{-1})\psi_z|_k[\alpha_i]$$

and extend linearly to $\mathscr{H}(\Gamma, G)$. Since ψ_z is invariant under Γ, this is well-defined. We get

Theorem 6.1.7. *The Hecke algebra $\mathscr{H}(\Gamma, G)$ acts on the space of modular forms of weight k and character χ for Γ.*

Proof. Let $\alpha \in G$ and $\alpha_i \in G$ for $i = 1, \ldots, h$ be as above. We can choose α_i such that $\chi(\alpha_i) = +1$ for all $i = 1, \ldots, h$. Let $\gamma \in \Gamma$. Then $\alpha_i\gamma = \gamma'\alpha_{i'}$ with $\chi(\gamma') = \chi(\gamma)$ because $\alpha_1, \ldots, \alpha_h$ is a full system of representatives of $\Gamma\alpha\Gamma$. Thus,

$$\psi_z|_k\Gamma\alpha\Gamma|_k[\gamma] = \sum_{i=1}^{h} \psi_z|_k[\alpha_i\gamma] = \sum_{i=1}^{h} \psi_z|_k[\gamma'\alpha_{i'}] = \chi(\gamma)\psi_z|_k\Gamma\alpha\Gamma,$$

where $i \mapsto i'$ is a bijection. □

Since scalar multiples of id transform trivial under the slash operator, again, it suffices to consider $\alpha \in G$ with $\alpha(L) \subset L$.

From now on we set $\Gamma = \mathrm{O}(L)^+$, $\Gamma_+ = \mathrm{SO}(L)^+$ and $G = \mathrm{GO}(L)^+$, $G_+ = \mathrm{GO}(L)^+_+$. Clearly $\mathcal{M}_k(\Gamma, \chi) \subset \mathcal{M}_k(\Gamma_+, 1)$ and so $\mathrm{End}_{\mathbb{C}}(\mathcal{M}_k(\Gamma, \chi)) \subset \mathrm{End}_{\mathbb{C}}(\mathcal{M}_k(\Gamma_+, 1))$. We find

Proposition 6.1.8. *The map* $\Gamma\alpha\Gamma \mapsto \Gamma\alpha\Gamma \cap G_+$ *defines an inclusion of Hecke algebras* $\mathscr{H}(\Gamma, G) \hookrightarrow \mathscr{H}(\Gamma_+, G_+)$ *and the diagram*

$$\begin{array}{ccc} \mathscr{H}(\Gamma, G) & \longrightarrow & \mathrm{End}_{\mathbb{C}}(\mathcal{M}_k(\Gamma, \chi)) \\ \big\downarrow & & \big\downarrow \\ \mathscr{H}(\Gamma_+, G_+) & \longrightarrow & \mathrm{End}_{\mathbb{C}}(\mathcal{M}_k(\Gamma_+, 1)) \end{array}$$

commutes.

Proof. Let $\alpha \in G$ and let

$$\Gamma\alpha\Gamma = \bigcup_{i=1}^{h} \Gamma\alpha_i$$

be a disjoint union. We can assume that $\det(\alpha_i) > 0$ for all $i \in \{1, \ldots, h\}$. Then

$$\Gamma\alpha\Gamma \cap G_+ = \bigcup_{i=1}^{h} \Gamma_+\alpha_i$$

is a disjoint union and

$$\sum_{i=1}^{h} \Gamma_+\alpha_i \in \mathscr{H}(\Gamma_+, G_+).$$

Now suppose that for $\alpha_1, \alpha_2 \in G$ we have $\Gamma\alpha_1\Gamma \cap G_+ = \Gamma\alpha_2\Gamma \cap G_+$. Let $\alpha \in \Gamma\alpha_1\Gamma$ be arbitrary. If $\alpha \in G_+$, then $\alpha \in \Gamma\alpha_1\Gamma \cap G_+ = \Gamma\alpha_2\Gamma \cap G_+ \subset \Gamma\alpha_2\Gamma$. If $\det(\alpha) < 0$, let $\sigma \in \Gamma$ be any element of determinant -1. Then $\sigma\alpha \in G_+$ and so $\sigma\alpha \in \Gamma\alpha_2\Gamma$. Hence, also $\alpha \in \Gamma\alpha_2\Gamma$. This implies $\Gamma\alpha_1\Gamma = \Gamma\alpha_2\Gamma$. □

Two elementary divisor theorems

We now want to prove elementary divisor theorems for the orthogonal groups $\mathrm{O}(L)^+$ and $\mathrm{SO}(L)^+$ similar to [30, Hilfssatz IV.1.12] for the symplectic group. This means that for each double coset in $\Gamma\backslash G/\Gamma$ (resp. $\Gamma_+\backslash G_+/\Gamma_+$) we want to find some canonical representative. By Proposition 6.1.5, for any $m \in \mathbb{Z}_{>0}$ the following maps are bijections:

$$\phi_1 : \Gamma\backslash G(m)/\Gamma \to \sim\backslash\{L(m) \subset L\},$$
$$\phi_1 : \Gamma_+\backslash G_+(m)/\Gamma_+ \to \sim_+\backslash\{L(m) \subset L\},$$

where $M_1 \sim M_2$ (resp. $M_1 \sim_+ M_2$) if and only if there exists a $\gamma \in \Gamma$ (resp. $\gamma \in \Gamma_+$) with $\gamma(M_1) = M_2$.

We define $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and for each p we fix a basis

$$(z_1, \ldots, z_k, z_k', \ldots, z_1')$$

of L_p with $k = n/2 + 1$ as we did in Section 5.1, i.e.

$$\mathrm{q}(z_i) = \mathrm{q}(z_i') = 0, \quad (z_i, z_i') = -1, \quad (z_i, z_j) = (z_i, z_j') = 0$$

for $i, j = 1, \ldots, k$ and $i \neq j$. Now let $m \in \mathbb{Z}_{>0}$ with prime decomposition $m = \prod p^{\nu_p(m)}$. Let $a_1 \mid \ldots \mid a_k \mid m$ with prime decompositions $a_i = \prod p^{\nu_p(a_i)}$. We define

$$L^{a_1,\ldots,a_k;m} := \bigcap (L_p^{\nu_p(a_1),\ldots,\nu_p(a_k);\nu_p(m)} \cap L_{\mathbb{Q}}).$$

Since for almost all primes we have $\nu_p(m) = 0$, this is well-defined and

$$L^{a_1,\ldots,a_k;m} \otimes_{\mathbb{Z}} \mathbb{Z}_p = L_p^{\nu_p(a_1),\ldots,\nu_p(a_k);\nu_p(m)}$$

by [41, Satz 21.5]. Recall the definition of $L_p^{\nu_p(a_1),\ldots,\nu_p(a_k);\nu_p(m)}$ on page 142.

To use the local results of Section 5.1 in the global setting we will need (cf. [17, Chapter 10, Section 7])

Theorem 6.1.9 (Strong Approximation Theorem for the Spin Group)**.** *Let L be an indefinite $\mathbb{Z}$-lattice of rank at least 3. Let $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$. For every prime p let $\mathcal{O}_p$ be a non-empty open subset of $\mathrm{SO}(V \otimes \mathbb{Q}_p)^+$ such that $\mathcal{O}_p = \mathrm{SO}(L \otimes \mathbb{Z}_p)^+$ for almost all p. Then there is an isometry $\sigma \in \mathrm{SO}(V)^+$ such that $\sigma \in \mathcal{O}_p$ for all p.*

Generally speaking, approximation theorems are in some sense a generalization of the Chinese remainder theorem to algebraic groups G over global fields k. We want to note that an analogous statement to Theorem 6.1.9 for the group $\mathrm{O}(V)^+$ does not hold.

Now we have

Theorem 6.1.10. *Let $M \subset L$ with $M \cong L(m)$ for some $m \in \mathbb{Z}_{>0}$. Then there exist unique integers $a_1 \mid \ldots \mid a_k \mid m$ with $a_i^2 \mid m$ for $i < k$ such that*

$$M \in \mathrm{SO}(L)^+ L^{a_1,\ldots,a_k;m}.$$

Proof. For a prime p by Proposition 5.1.10 there exist integers $l_1 \leq \ldots \leq l_k \leq \nu_p(m)/2$ such that

$$M_p \in \mathrm{O}(L_p)^+ L_p^{l_1,\ldots,l_k;\nu_p(m)}.$$

Now let

$$\mathcal{O}_p' := \{\gamma \in \mathrm{O}(L_p)^+ \mid M_p = \gamma(L_p^{l_1,\ldots,l_k;\nu_p(m)})\}.$$

If $l_k < \nu_p(m)/2$, then by Lemma 5.1.11 the determinant of a $\gamma \in \mathcal{O}_p'$ does not depend on the choice of γ. If $\det(\gamma) = -1$, we replace l_k by $l - l_k$. Then any element in $\mathcal{O}_p'$ is of the form $\gamma\sigma_{z_k - z_k'}$, with γ as before and $\mathcal{O}_p' \subset \mathrm{SO}(L_p)^+$. If $l_k = \nu_p(m)/2$, then

$\sigma_{z_k - z'_k}$ fixes $L_p^{l_1,\dots,l_k;\nu_p(m)}$ and so $\mathcal{O}'_p$ contains elements of determinant 1 and -1. In either case we have

$$\emptyset \neq \mathcal{O}_p := \mathcal{O}'_p \cap \mathrm{SO}(L_p)^+ \subset \mathrm{SO}(L_p)^+$$

for all p and $\mathcal{O}_p = \mathrm{SO}(L_p)^+$ whenever $p \nmid m$. Note that $\mathrm{SO}(L_p)^+$ and $\mathrm{SO}(M_p)^+$ are open since $\mathbb{Z}_p$ is open in $\mathbb{Q}_p$ and hence,

$$\mathcal{O}_p = \mathrm{SO}(L_p)^+ \cap \mathrm{SO}(M_p)^+ \gamma$$

is also open, where γ is any element in $\mathcal{O}_p$. By the Strong Approximation Theorem for the Spin Group there exists a $\gamma \in \mathrm{SO}(L_\mathbb{Q})^+$ such that $\gamma \in \mathcal{O}_p$ for all primes p. By [41, Satz 21.5] we then have

$$\begin{aligned} M &= \bigcap (M_p \cap L_\mathbb{Q}) \\ &= \bigcap (\gamma(L_p^{l_1,\dots,l_k;\nu_p(m)}) \cap L_\mathbb{Q}) \\ &= \gamma \left(\bigcap (L_p^{l_1,\dots,l_k;\nu_p(m)} \cap L_\mathbb{Q}) \right) \\ &= \gamma(L^{a_1,\dots,a_k;m}) \end{aligned}$$

with $a_i = \prod p^{l_i}$. Note that $l_i = 0$ for almost all p. An analogous calculation shows that $\gamma \in \mathrm{SO}(L)^+$. The uniqueness of the integers a_i follows from the uniqueness of the l_i. □

We can deduce the corresponding theorem for $\mathrm{O}(L)^+$ from the theorem for $\mathrm{SO}(L)^+$. We will need

Lemma 6.1.11. *Let $M \subset L$ with $M \cong L(m)$ and let $a_1 \mid \dots \mid a_k \mid m$ be the integers from Theorem 6.1.10 corresponding to M. Furthermore, let $\sigma \in \mathrm{O}(L)^+$ be any element of determinant -1. Then*

$$\sigma(M) \in \mathrm{SO}(L)^+ L^{a_1,\dots,a_{k-1},m/a_k;m}.$$

Proof. Let p be any prime and let $\mathcal{O}_p$ be as in the proof of Theorem 6.1.10 corresponding to M and let $\mathcal{O}_p^\sigma$ be the analogous set corresponding to $\sigma(M)$. We write $l = \nu_p(m)$ and $l_i = \nu_p(a_i)$ for $i = 1, \dots, k$. Then

$$\emptyset \neq \sigma \mathcal{O}_p \sigma_{z_k - z'_k} = \{\gamma \in \mathrm{SO}(L_p)^+ \mid \sigma(M_p) = \gamma(L_p^{l_1,\dots,l_{k-1},l-l_k;l})\}$$

since $\sigma_{z_k - z'_k}(L_p^{l_1,\dots,l_{k-1},l-l_k;l}) = L_p^{l_1,\dots,l_k;l}$ and so

$$\mathcal{O}_p^\sigma = \sigma \mathcal{O}_p \sigma_{z_k - z'_k}.$$

But this implies that the invariants corresponding to $\sigma(M)$ are $a_1, \dots, a_{k-1}, m/a_k$. □

Theorem 6.1.12. *Let $M \subset L$ with $M \cong L(m)$ for some $m \in \mathbb{Z}_{>0}$. Then there exist unique integers $a_1 \mid \ldots \mid a_k \mid m$ with $a_i^2 \mid m$ for $i < k$ and $a_k \leq \sqrt{m}$ such that*

$$M \in \mathrm{O}(L)^+ L^{a_1,\ldots,a_k;m}.$$

Proof. Let $a_1 \mid \ldots \mid a_k \mid m$ be the unique integers from Theorem 6.1.10 corresponding to M and let $\sigma \in \mathrm{O}(L)^+$ be any element of determinant -1. By the previous proposition $M \in \sigma\,\mathrm{SO}(L)^+ L^{a_1,\ldots,a_{k-1},m/a_k;m}$ and in particular

$$M \in \mathrm{O}(L)^+ L^{a_1,\ldots,a_k;m} = \mathrm{O}(L)^+ L^{a_1,\ldots,a_{k-1},m/a_k;m}.$$

The set $\{a_k, m/a_k\}$ contains exactly one element $\leq \sqrt{m}$. This proves the theorem. $\square$

Note that in the case of $\mathrm{O}(L)^+$ the invariant a_k depends on the choice of basis for L_p. Swapping the last two basis vectors of L_p for some prime p while keeping the basis for all $q \neq p$ changes $\nu_p(a_k)$ to $\nu_p(m) - \nu_p(a_k)$ while keeping all $\nu_q(a_k)$.

From the elementary divisor theorems follows

Theorem 6.1.13. *The Hecke algebras $\mathscr{H}(\Gamma_+, G_+)$ and $\mathscr{H}(\Gamma, G)$ are commutative.*

Proof. We prove the commutativity of $\mathscr{H}(\Gamma_+, G_+)$. For $\mathscr{H}(\Gamma, G)$ it then follows from Proposition 6.1.8. Let α be in $G_+(m)$ for some positive integer m. Let $a_1, \ldots, a_k$ be the integers from Theorem 6.1.10 corresponding to $\alpha(L)$. Then there exists a $\gamma \in \Gamma_+$ such that

$$\gamma\alpha(L) = L^{a_1,\ldots,a_k;m}.$$

This implies that for all primes p there exists a $\beta \in \mathrm{O}(L_p)$ such that

$$\gamma\alpha\beta(z_i) = p^{\nu_p(a_i)} z_i$$
$$\gamma\alpha\beta(z_i') = p^{\nu_p(m)-\nu_p(a_i)} z_i'.$$

If σ is the element from Lemma 5.1.5 with $z = z_1$, $z' = z_1'$ and $u = \mathrm{spin}(\beta)$, we can replace γ by $\sigma\gamma$ and β by $\beta\sigma^{-1}$ and therefore assume that $\beta \in \mathrm{O}(L_p)^+$. It is clear that $\det(\gamma\alpha\beta) = p^{\nu_p(m)k}$ so that $\det(\beta) = +1$. Now

$$\beta^{-1} p^{\nu_p(m)} \alpha^{-1}\gamma^{-1}(z_i) = p^{\nu_p(m)-\nu_p(a_i)} z_i$$
$$\beta^{-1} p^{\nu_p(m)} \alpha^{-1}\gamma^{-1}(z_i') = p^{\nu_p(a_i)} z_i'.$$

Note that $\sigma := \sigma_{z_1 - z_1'} \ldots \sigma_{z_k - z_k'} \in \mathrm{SO}(L_p)^+$ and so

$$\sigma^{-1}\beta^{-1} p^{\nu_p(m)} \alpha^{-1}\gamma^{-1}\sigma(z_i) = p^{\nu_p(a_i)} z_i$$
$$\sigma^{-1}\beta^{-1} p^{\nu_p(m)} \alpha^{-1}\gamma^{-1}\sigma(z_i') = p^{\nu_p(m)-\nu_p(a_i)} z_i',$$

i.e.

$$p^{\nu_p(m)}\alpha^{-1}(L_p) \in \mathrm{SO}(L_p)^+ L_p^{\nu_p(a_1),\dots,\nu_p(a_k);\nu_p(m)}.$$

Again, by the Strong Approximation Theorem for the Spin Group this implies

$$m\alpha^{-1}(L) \in \mathrm{SO}(L)^+ L^{a_1,\dots,a_k;m}.$$

Hence, $\alpha \mapsto m\alpha^{-1}$ is an antihomomorphism with the property $\Gamma_+\alpha\Gamma_+ = \Gamma_+ m\alpha^{-1}\Gamma_+$ and so by Theorem 6.1.2 $\mathscr{H}(\Gamma_+, G_+)$ is commutative. □

Local Hecke algebras

We define *local Hecke algebras*. Let p be a prime and let

$$G_p = \mathrm{GO}\left(L \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right]\right)^+ \quad \text{and} \quad G_{+,p} = \mathrm{GO}\left(L \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right]\right)^+_+.$$

Since $G_p \subset G$ and $G_{+,p} \subset G_+$, we have $\mathscr{H}(\Gamma, G_p) \subset \mathscr{H}(\Gamma, G)$ and $\mathscr{H}(\Gamma_+, G_{+,p}) \subset \mathscr{H}(\Gamma_+, G_+)$, hence

$$\prod_p{}' \mathscr{H}(\Gamma, G_p) \subset \mathscr{H}(\Gamma, G) \quad \text{and} \quad \prod_p{}' \mathscr{H}(\Gamma_+, G_{+,p}) \subset \mathscr{H}(\Gamma_+, G_+),$$

where $\prod'_{i\in I} A_i \coloneqq \{\sum_{j=1}^r \prod_{i\in I} a_{j,i} \mid a_{j,i} \in A_i,\ a_{j,i} = 1 \text{ for almost all } i \in I\}$. Note that $G_p(p) = G(p)$ and $G_{+,p}(p) = G_+(p)$. By Theorem 6.1.12 for primes $p \neq q$ we have $|\Gamma\backslash G(p)/\Gamma| = |\Gamma\backslash G(q)/\Gamma| = 1$, but $|\Gamma\backslash G(pq)/\Gamma| = 2$. And so clearly

$$\prod_p{}' \mathscr{H}(\Gamma, G_p) \neq \mathscr{H}(\Gamma, G).$$

We show that for $\mathrm{SO}(L)^+$ the local Hecke algebras generate the global one.

Proposition 6.1.14. $\prod'_p \mathscr{H}(\Gamma_+, G_{+,p}) = \mathscr{H}(\Gamma_+, G_+)$.

Proof. Note that for any $N = \prod_p p^{\nu_p(N)} \in \mathbb{Q}$ the double coset $\Gamma_+ p^{\nu_p(N)}\Gamma_+ = \Gamma_+ p^{\nu_p(N)} \in \mathscr{H}(\Gamma_+, G_{+,p})$ consists of only one right coset and so

$$\prod_p (\Gamma_+ p^{\nu_p(N)}\Gamma_+) = \Gamma_+ N = \Gamma_+ N \Gamma_+.$$

Therefore, it suffices to show that for any $\alpha \in G_+(m)$ the double coset $\Gamma_+\alpha\Gamma_+$ is in $\prod'_p \mathscr{H}(\Gamma_+, G_{+,p})$. Let $a_1 \mid \dots \mid a_k$ be the invariants from Theorem 6.1.10 corresponding to α. For $p \mid m$ let $\alpha_p \in G_{+,p}$ be any element with the property

$$\alpha_p(L) \in \Gamma_+ L^{p^{\nu_p(a_1)},\dots,p^{\nu_p(a_k)};p^{\nu_p(m)}}. \tag{6.1.1}$$

Let $\alpha_{p,q}$ be the extension of α_p to L_q for a prime q. If $q \neq p$, then $\alpha_{p,q} \in \mathrm{GO}(L_q)_+$ and we can write $\alpha_{p,q} = \gamma\sigma$ with $\gamma \in \mathrm{SO}(L_q)^+$ and $\sigma \in \mathrm{GO}(L_q)_+$ is given by

$$\begin{aligned} z_i &\mapsto a_i z_i \\ z_i' &\mapsto b_i z_i' \end{aligned}$$

with $a_i b_i = p^{\nu_p(m)}$ for $i = 1, \ldots, k$. Note that $a_i, b_i \in \mathbb{Z}_q^\times$ and so

$$\left(\prod_{p|m} \alpha_{p,q}\right)(L_q) \in \mathrm{SO}(L_q)^+ L_q^{\nu_q(a_1),\ldots,\nu_q(a_k);\nu_q(m)}.$$

The product is understood to be ordered by the primes in the obvious way. As in the proof of Theorem 6.1.10, by the Strong Approximation Theorem for the Spin Group we have

$$\left(\prod_{p|m} \alpha_p\right)(L) \in \mathrm{SO}(L)^+ \alpha(L)$$

so that

$$\prod_{p|m} \alpha_p \in \Gamma_+ \alpha \Gamma_+.$$

Since the elements in $\Gamma_+\alpha_p\Gamma_+$ are precisely those that satisfy (6.1.1) and α_p was chosen arbitrarily with this property, it follows that

$$\prod_{p|m} \Gamma_+\alpha_p\Gamma_+ = r\Gamma_+\alpha\Gamma_+$$

for some $r \in \mathbb{Z}_{>0}$. We show that actually $r = 1$:

Let

$$\Gamma_+\alpha_p\Gamma_+ = \bigcup_{i=1}^{h_p} \Gamma_+\alpha_{p,i}$$

for $p \mid m$ and assume that for some $i_p, j_p \in \{1, \ldots, h_p\}$ and $\gamma \in \Gamma_+$ we have

$$\prod_{p|m} \alpha_{p,i_p} = \gamma \prod_{p|m} \alpha_{p,j_p}.$$

Then

$$\alpha_{q,i_q}\alpha_{q,j_q}^{-1} = \left(\prod_{p|m'} \alpha_{p,i_p}\right)^{-1} \gamma \prod_{p|m'} \alpha_{p,j_p},$$

where q is the largest prime dividing m and $m = q^k m'$ with $(q, m') = 1$. Now the left-hand side of this equation is in $\mathrm{SO}\left(L \otimes \mathbb{Z}\left[\frac{1}{q}\right]\right)^+$, whereas the right-hand side is in $\mathrm{SO}\left(L \otimes \mathbb{Z}\left[\frac{1}{m'}\right]\right)^+$. The intersection of those is just $\mathrm{SO}(L)^+$ and so $i_q = j_q$. By induction on the number of prime factors of m, the claim follows. □

6.2 The Hecke operator $\mathcal{T}(p)$

Let $m \in \mathbb{Z}_{>0}$ and let $a_1 \mid \ldots \mid a_{n/2+1} \mid m$ with $a_i^2 \mid m$ for $i < n/2+1$. We define the Hecke operator

$$\mathcal{T}^{a_1,\ldots,a_{n/2+1}}(m) := \Gamma_+\alpha\Gamma_+ \in \mathscr{H}(\Gamma_+, G_+),$$

where $\alpha \in G_+(m)$ such that $\alpha(L) = L^{a_1,\ldots,a_{n/2+1};m}$.

Let from now on throughout p be a fixed prime. It follows from Theorems 6.1.10 and 6.1.12 that

$$G(p) = \Gamma\alpha_1\Gamma = \Gamma_+\alpha_1\Gamma_+ \cup \Gamma_+\alpha_2\Gamma_+$$

for $\alpha_1, \alpha_2 \in G_+(p)$ such that $\alpha_1(L) = L^{1,\ldots,1;p}$ and $\alpha_2(L) = L^{1,\ldots,1,p;p}$. We set

$$\mathcal{T}(p) = \mathcal{T}^{1,\ldots,1}(p) + \mathcal{T}^{1,\ldots,1,p}(p) \in \mathscr{H}(\Gamma, G),$$

i.e.

$$\psi_z|_k\mathcal{T}(p) = \sum_{\alpha\in\Gamma\backslash G(p)} \chi(\alpha^{-1})\psi_z|_k[\alpha].$$

We now want to describe a decomposition of $G(p)$ into right cosets. It follows from Lemma 5.1.4 that both $X_{\mathrm{SO}(L)^+}$ and $X_{\mathrm{O}(L)^+}$ contains only one 0-dimensional cusp, so let from now on $z, z' \in L$ be fixed and set $K = L \cap z^\perp \cap z'^\perp$. We choose primitive isotropic elements $w_{+j}, w_{-j} \in K$ with $(w_{+j}, w_{-j}) = -1$ and set $L_j = K \cap w_{+j}^\perp \cap w_{-j}^\perp$ such that $L_1, \ldots, L_h \in \mathit{II}_{n-2,0}$ is a complete system of representatives, which is possible since L splits two hyperbolic planes. By possibly replacing $w_{\pm j}$ with $-w_{\pm j}$ we can assume that $w_{+j} + w_{-j} \in \mathcal{C}$. We also set $w = w_{+1}$, $w' = w_{-1}$.

For any $M \in \{L_j(p) \subset L_1\}$ we choose an embedding $\beta : L_j \to M$ by which we mean that β is an isomorphism of $\mathbb{Z}$-modules with $(\beta(\lambda), \beta(\mu)) = p(\lambda, \mu)$ for all $\lambda, \mu \in L_j$. We define an element $\alpha_M^{0,0} \in G_+(p)$ by

$$\begin{aligned} z &\mapsto pz \\ z' &\mapsto z' \\ w &\mapsto pw_{\epsilon j} \\ w' &\mapsto w_{-\epsilon j} \\ \lambda &\mapsto \beta^{-1}(p\lambda) \text{ for } \lambda \in L_1, \end{aligned}$$

where $\epsilon = \pm 1$ is chosen such that $\det(\alpha_M^{0,0}) > 0$ and depends on β. Note that for $j = 1$ we can view β as an element in $\mathrm{GO}(L_1 \otimes_\mathbb{Z} \mathbb{Q})$ and then $\epsilon p^{n/2-1} = \det(\beta)$. Indeed $\mathrm{spin}(\alpha_M^{0,0}) = 1$ since $\alpha_M^{0,0}$ maps $w + w'$ to $pw_{\epsilon j} + w_{-\epsilon j}$, which are both in $\mathcal{C}$.

Similarly, we define $\alpha_M^{0,1} \in G_+(p)$ by

$$\begin{aligned} z &\mapsto pz \\ z' &\mapsto z' \\ w &\mapsto w_{\epsilon j} \\ w' &\mapsto pw_{-\epsilon j} \\ \lambda &\mapsto \beta^{-1}(p\lambda) \text{ for } \lambda \in L_1. \end{aligned}$$

We set $L_M^{0,1} := \beta^{-1}(pL_1) \in \{L_1(p) \subset L_j\}$ and let $\mu \in L_M^{0,1}$. Then $E_{p^{-1}\mu}^{w_{\epsilon j}} \alpha_M^{0,1} \in G_+(p)$ because by Proposition 5.1.3 we have

$$E_{p^{-1}\mu}^{w_{\epsilon j}} \alpha_M^{0,1} = \alpha_M^{0,1} E_{\mu'}^{w_{\epsilon j}}$$

with $\alpha_M^{0,1}(\mu') = \mu$ and $E_{\mu'}^{w} \in \Gamma_+$ since $\mu' \in L_1$.
Furthermore, we define $\alpha_M^{1,0} \in G_+(p)$ by

$$\begin{aligned} z &\mapsto z \\ z' &\mapsto pz' \\ w &\mapsto pw_{\epsilon j} \\ w' &\mapsto w_{-\epsilon j} \\ \lambda &\mapsto \beta^{-1}(p\lambda) \text{ for } \lambda \in L_1 \end{aligned}$$

and $L_M^{1,0} := \beta^{-1}(pL_1) \oplus p\mathbb{Z}w_{\epsilon j} \oplus \mathbb{Z}w_{-\epsilon j}$. Then for $\nu \in L_M^{1,0}$ also

$$E_{p^{-1}\nu}^{z} \alpha_M^{1,0} = \alpha_M^{1,0} E_{\nu'}^{z} \in G_+(p)$$

with $\alpha_M^{1,0}(\nu') = \nu$.
Finally, we define $\alpha_M^{1,1} \in G_+(p)$ by

$$\begin{aligned} z &\mapsto z \\ z' &\mapsto pz' \\ w &\mapsto w_{\epsilon j} \\ w' &\mapsto pw_{-\epsilon j} \\ \lambda &\mapsto \beta^{-1}(p\lambda) \text{ for } \lambda \in L_1. \end{aligned}$$

and $L_M^{1,1} := \beta^{-1}(pL_1) \oplus \mathbb{Z}w_{\epsilon j} \oplus p\mathbb{Z}w_{-\epsilon j}$. Then for $\nu \in L_M^{1,1}$ and $\mu \in L_M^{0,1}$ also

$$E_{p^{-1}\nu}^{z} E_{p^{-1}\mu}^{w_{\epsilon j}} \alpha_M^{1,1} = \alpha_M^{1,1} E_{\nu'}^{z} E_{\mu'}^{w_{\epsilon j}} \in G_+(p)$$

with $\alpha_M^{1,1}(\nu') = \nu$ and $\alpha_M^{1,1}(\mu') = \mu$.

Note that all of the above elements depend on the choice of the embedding β, which we have omitted from the notation. However, as we will see later, a different choice of β corresponds to a multiplication by some element in Γ from the left.

We need to understand for which μ and ν the above elements are in the same right coset modulo Γ. For $E^{w_{\epsilon j}}_{p^{-1}\mu}\alpha^{0,1}_M$ and $E^{z}_{p^{-1}\nu}\alpha^{1,0}_M$ this is not so difficult as we will see in the next lemma. However, for $E^{z}_{p^{-1}\nu}E^{w_{\epsilon j}}_{p^{-1}\mu}\alpha^{1,1}_M$ it is more complicated. We define an equivalence relation $\sim_M$ on $L^{1,1}_M \times L^{0,1}_M$ in the following way:
Let $(\nu,\mu),(\nu',\mu') \in L^{1,1}_M \times L^{0,1}_M$. Then we say that $(\nu,\mu) \sim_M (\nu',\mu')$ if and only if

$$\mu-\mu' \in pL_j \quad \text{and} \quad \nu-\nu' \in pK + \frac{(\nu,\mu-\mu')}{p}w_{\epsilon j}.$$

Note that

$$(\nu,\mu-\mu') = (\nu',\mu-\mu') + (\nu-\nu',\mu-\mu').$$

And so if $(\nu,\mu) \sim_M (\nu',\mu')$, then $(\nu-\nu',\mu-\mu') \in p^2\mathbb{Z}$, which implies that also

$$\nu-\nu' \in pK + \frac{(\nu',\mu-\mu')}{p}w_{\epsilon j}.$$

From this follows symmetry and transitivity of the relation. Reflexivity is clear. Then we have

Lemma 6.2.1. *For $j = 1,\dots,h$ and $M \in \{L_j(p) \subset L_1\}$ the sets*

$$\begin{aligned}
G(p)^{0,0}_M &:= \Gamma\alpha^{0,0}_M,\\
G(p)^{0,1}_M &:= \bigcup_{\mu \in L^{0,1}_M/pL_j} \Gamma E^{w_{\epsilon j}}_{p^{-1}\mu}\alpha^{0,1}_M,\\
G(p)^{1,0}_M &:= \bigcup_{\nu \in L^{1,0}_M/pK} \Gamma E^{z}_{p^{-1}\nu}\alpha^{1,0}_M,\\
G(p)^{1,1}_M &:= \bigcup_{(\nu,\mu)\in L^{1,1}_M\times L^{0,1}_M/\sim_M} \Gamma E^{z}_{p^{-1}\nu}E^{w_{\epsilon j}}_{p^{-1}\mu}\alpha^{1,1}_M.
\end{aligned}$$

are disjoint unions.

Proof. We need to show that these unions are well-defined and disjoint. If $\mu,\mu' \in L^{0,1}_M$, then

$$E^{w_{\epsilon j}}_{p^{-1}(\mu+\mu')}\alpha^{0,1}_M = E^{w_{\epsilon j}}_{p^{-1}\mu'}E^{w_{\epsilon j}}_{p^{-1}\mu}\alpha^{0,1}_M$$

and $E^{w_{\epsilon j}}_{p^{-1}\mu'} \in \Gamma$ if and only if $\mu' \in pL_j$. Therefore, the first union is well-defined and disjoint. We proceed analogously for the second one. For the third let $\nu,\nu' \in L^{1,1}_M$ and $\mu,\mu' \in L^{0,1}_M$. We have

$$\begin{aligned}
E^{z}_{p^{-1}(\nu+\nu')}E^{w_{\epsilon j}}_{p^{-1}(\mu+\mu')} &= E^{z}_{p^{-1}\nu'}E^{z}_{p^{-1}\nu}E^{w_{\epsilon j}}_{p^{-1}\mu'}E^{w_{\epsilon j}}_{p^{-1}\mu}\\
&= E^{z}_{p^{-1}\nu'}E^{\widetilde{w}}_{p^{-1}\widetilde{\mu}}E^{z}_{p^{-1}\nu}E^{w_{\epsilon j}}_{p^{-1}\mu}
\end{aligned}$$

with

$$\widetilde{w} = E^z_{p^{-1}\nu}(w_{\epsilon j}) = w_{\epsilon j} + \frac{(w_{\epsilon j}, \nu)}{p} z \in L$$
$$\widetilde{\mu} = E^z_{p^{-1}\nu}(\mu') = \mu' + \frac{(\mu', \nu)}{p} z \in L.$$

If $\mu' \in pL_j$ and $\nu' \in pK + p^{-1}(\nu, \mu')w_{\epsilon j}$, we write $\nu' = p\widetilde{\nu} + aw_{\epsilon j}$ with $\widetilde{\nu} \in K$ and $a = p^{-1}(\nu, \mu') \in \mathbb{Z}$. Then

$$E^z_{p^{-1}\nu'} E^{\widetilde{w}}_{p^{-1}\widetilde{\mu}} = E^z_{\widetilde{\nu}} E^z_{p^{-1}aw_{\epsilon j}} E^{\widetilde{w}}_{p^{-1}az} E^{\widetilde{w}}_{p^{-1}\mu'},$$

where $E^z_{\widetilde{\nu}}$ and $E^{\widetilde{w}}_{p^{-1}\mu'}$ are in Γ and one verifies that $E^z_{p^{-1}aw_{\epsilon j}} E^{\widetilde{w}}_{p^{-1}az} = \mathrm{id}$. Hence, the union is well-defined. To show that it is disjoint assume that

$$E^z_{p^{-1}\nu'} E^{\widetilde{w}}_{p^{-1}\widetilde{\mu}} \in \Gamma.$$

For $\lambda \in L_j$ we have

$$\begin{aligned} E^z_{p^{-1}\nu'} E^{\widetilde{w}}_{p^{-1}\widetilde{\mu}}(\lambda) &= E^z_{p^{-1}\nu'} \left(\lambda + \frac{(\lambda, \mu')}{p} \left(w_{\epsilon j} + \frac{(w_{\epsilon j}, \nu)}{p} z \right) \right) \\ &= \lambda + \frac{(\lambda, \nu')}{p} z + \frac{(\lambda, \mu')}{p} \left(w_{\epsilon j} + \frac{(w_{\epsilon j}, \nu)}{p} z \right) + \frac{(\lambda, \mu')(w_{\epsilon j}, \nu')}{p^2} z. \end{aligned}$$

This implies $(\lambda, \mu') \in p\mathbb{Z}$ and then also $(\lambda, \nu') \in p\mathbb{Z}$. Since L_j is unimodular, this is equivalent to $\mu' \in pL_j$ and $\nu' \in pK + aw_{\epsilon j}$ for some $a \in \mathbb{Z}$. Now we have

$$\begin{aligned} E^z_{p^{-1}\nu'} E^{\widetilde{w}}_{p^{-1}\widetilde{\mu}}(w_{-\epsilon j}) &= E^z_{p^{-1}\nu'} \left(w_{-\epsilon j} + p^{-1}\mu' + \frac{(\mu', \nu)}{p^2} z + \frac{\mathrm{q}(\mu')}{p^2} \left(w_{\epsilon j} + \frac{(w_{\epsilon j}, \nu)}{p} z \right) \right) \\ &= w_{-\epsilon j} + \frac{(w_{-\epsilon j}, \nu')}{p} z + p^{-1}\mu' + \frac{(\mu', \nu')}{p^2} z \\ &\quad + \frac{(\mu', \nu)}{p^2} z + \frac{\mathrm{q}(\mu')}{p^2} \left(w_{\epsilon j} + \frac{(w_{\epsilon j}, \nu)}{p} z \right) + \frac{\mathrm{q}(\mu')}{p^3} (w_{\epsilon j}, \nu') z. \end{aligned}$$

We already know that $(\mu', \nu') \in p^2\mathbb{Z}$, $\mathrm{q}(\mu')(w_{\epsilon j}, \nu) \in p^3\mathbb{Z}$ and $\mathrm{q}(\mu')(w_{\epsilon j}, \nu') \in p^3\mathbb{Z}$. Therefore, we must have

$$(w_{-\epsilon j}, \nu') + \frac{(\mu', \nu)}{p} \in p\mathbb{Z},$$

i.e. $a = p^{-1}(\mu', \nu) \bmod p$. Since $pw_{\epsilon j} \in pK$, we can assume $a = p^{-1}(\mu', \nu)$. Hence, $(\nu + \nu', \mu + \mu') \sim_M (\nu, \mu)$ and the union is disjoint. □

An easy method for finding a full system of representatives for $\sim_M$ is given in

Lemma 6.2.2. *Let $R \subset L^{1,1}_M$ and $S \subset L^{0,1}_M$ be full systems of representatives for $L^{1,1}_M/pK$ and $L^{0,1}_M/pL_j$ respectively. Then $R \times S$ is a full system of representatives for $L^{1,1}_M \times L^{0,1}_M / \sim_M$.*

Proof. Let $(\nu,\mu),(\nu',\mu')\in R\times S$ with $(\nu,\mu)\sim_M(\nu',\mu')$. Then $\mu-\mu'\in pL_j$ and hence $\mu-\mu'=0$. But then also $\nu-\nu'\in pK$, so that $\nu-\nu'=0$ and $(\nu,\mu)=(\nu',\mu')$. Let on the other hand $(\nu,\mu)\in L_M^{1,1}\times L_M^{0,1}$ be arbitrary. Then there exists a $\mu'\in S$ such that $\mu\in\mu'+pL_j$. Now there exists a $\nu'\in R$ such that

$$\nu-\frac{(\nu,\mu-\mu')}{p}w_{\epsilon j}\in\nu'+pK.$$

Then $(\nu,\mu)\sim_M(\nu',\mu')$. □

We have seen that whenever we have systems of representatives for $L_M^{1,1}/pK$ and for $L_M^{0,1}/pL_j$, this defines a bijection from $L_M^{1,1}/pK\times L_M^{0,1}/pL_j$ to $L_M^{1,1}\times L_M^{0,1}/\sim_M$. However, note that $L_M^{1,1}/pK\times L_M^{0,1}/pL_j\neq L_M^{1,1}\times L_M^{0,1}/\sim_M$.

It is not difficult to see that

$$|L_M^{0,1}/pL_j|=p^{n/2-1}\quad\text{and}\quad|L_M^{1,0}/pK|=p^{n/2}$$

and using the previous lemma also

$$|L_M^{1,1}\times L_M^{0,1}/\sim_M|=p^{n-1}.$$

Now we can finally describe the decomposition of $G(p)$ into right cosets.

Proposition 6.2.3. *The union*

$$G(p)=\bigcup_{j=1}^{h}\bigcup_{M\in\{L_j(p)\subset L_1\}}\left(G(p)_M^{0,0}\cup G(p)_M^{0,1}\cup G(p)_M^{1,0}\cup G(p)_M^{1,1}\right)$$

is disjoint.

Proof. First we show that every $\alpha\in G(p)$ is in one of the above sets:
Let $\alpha(z)=(p/a)v$ for some primitive isotropic $v\in L$ and $a\in\{1,p\}$. By Lemma 5.1.4 there exists a $\gamma_1\in\Gamma_+$ with $\gamma_1(v)=z$ and so $\gamma_1\alpha(z)=(p/a)z$. This implies

$$\gamma_1\alpha(z')=az'+\frac{\mathrm{q}(\nu')}{a}z+\nu'$$

for some $\nu'\in K$ with $\mathrm{q}(\nu')\in a\mathbb{Z}$.
Now let $\gamma_1\alpha(w)=xz+yz'+\lambda$ for some $\lambda\in K$. Then

$$0=p(w,z)=(\gamma_1\alpha(w),\gamma_1\alpha(z))=-\frac{yp}{a},$$

so that $y=0$ and

$$0=p\,\mathrm{q}(w)=\mathrm{q}(\gamma_1\alpha(w))=\mathrm{q}(\lambda).$$

Let $\lambda = (p/b)u$ with $u \in K$ primitive isotropic and $b \in \{1, p\}$. Since $L_1, \ldots, L_h$ is a full system of representatives for $II_{n-2,0}$, there exits some $j = 1, \ldots, h$ such that there exists a $\gamma_2 \in \mathrm{O}(L)$ with

$$\begin{aligned} z &\mapsto z \\ z' &\mapsto z' \\ u &\mapsto w_{\epsilon j}. \end{aligned}$$

Then

$$\begin{aligned} \gamma_2\gamma_1\alpha(z) &= \frac{p}{a} z \\ \gamma_2\gamma_1\alpha(z') &= az' + \frac{\mathrm{q}(\gamma_2(\nu'))}{a} z + \gamma_2(\nu') \\ \gamma_2\gamma_1\alpha(w) &= \frac{p}{b} w_{\epsilon j} + \frac{p}{ab}(w_{\epsilon j}, \gamma_2(\nu'))z \\ \gamma_2\gamma_1\alpha(w') &= bw_{-\epsilon j} + \frac{\mathrm{q}(\mu')}{b} w_{\epsilon j} + xz + \mu' \end{aligned}$$

for $\mu' \in L_j$ with $\mathrm{q}(\mu') \in b\mathbb{Z}$ and

$$x = \frac{1}{a}\left(b(w_{-\epsilon j}, \gamma_2(\nu')) + \frac{\mathrm{q}(\mu')}{b}(w_{\epsilon j}, \gamma_2(\nu')) + (\mu', \gamma_2(\nu')) \right).$$

Finally, we find for $\lambda \in L_1$ that

$$\gamma_2\gamma_1\alpha(\lambda) = \lambda' + yz + \frac{(\lambda', \mu')}{b} w_{\epsilon j},$$

where $\lambda' \in L_j$ and

$$y = \frac{1}{a}\left((\lambda', \gamma_2(\nu')) + \frac{(\lambda', \mu')(w_{\epsilon j}, \gamma_2(\nu'))}{b} \right).$$

In fact the map $\delta : L_1 \to L_j$ given by $\delta(\lambda) = \lambda'$ is an embedding from L_1 to M' for some $M' \in \{L_1(p) \subset L_j\}$. We set $\beta' := p\delta^{-1}$. Then β' defines an embedding $\beta' : L_j \to M$ for some $M \in \{L_j(p) \subset L_1\}$. Now let $\beta : L_j \to M$ be the embedding chosen for M in the beginning. We define $\gamma_3 \in \Gamma$ by

$$\begin{aligned} z &\mapsto z \\ z' &\mapsto z' \\ w_{\epsilon j} &\mapsto w_{\epsilon j} \\ w_{-\epsilon j} &\mapsto w_{-\epsilon j} \\ \lambda &\mapsto \beta^{-1}\beta'(\lambda) \quad \text{for } \lambda \in L_j. \end{aligned}$$

Then

$$\begin{aligned}\gamma_3\gamma_2\gamma_1\alpha(\lambda) &= \beta^{-1}(p\lambda) + yz + \frac{(\beta'^{-1}(p\lambda), \mu')}{b} w_{\epsilon j} \\ &= \beta^{-1}(p\lambda) + yz + \frac{(\beta^{-1}(p\lambda), \gamma_3(\mu'))}{b} w_{\epsilon j}\end{aligned}$$

with

$$\begin{aligned}y &= \frac{1}{a}\left((\beta'^{-1}(p\lambda), \gamma_2(\nu')) + \frac{(\beta'^{-1}(p\lambda), \mu')(w_{\epsilon j}, \gamma_2(\nu'))}{b} \right) \\ &= \frac{1}{a}\left((\beta^{-1}(p\lambda), \gamma_3\gamma_2(\nu')) + \frac{(\beta^{-1}(p\lambda), \gamma_3(\mu'))(w_{\epsilon j}, \gamma_3\gamma_2(\nu'))}{b} \right).\end{aligned}$$

Therefore, the element $\gamma_3\gamma_2\gamma_1\alpha$ is given by

$$\begin{aligned}z &\mapsto \frac{p}{a} z \\ z' &\mapsto az' + \frac{\mathrm{q}(\nu)}{a} z + \nu \\ w &\mapsto \frac{p}{b} w_{\epsilon j} + \frac{p}{ab}(w_{\epsilon j}, \nu) z \\ w' &\mapsto b w_{-\epsilon j} + \frac{\mathrm{q}(\mu)}{b} w_{\epsilon j} + xz + \mu \\ \lambda &\mapsto \beta^{-1}(p\lambda) + yz + \frac{(\beta^{-1}(p\lambda), \mu)}{b} w_{\epsilon j} \quad \text{for } \lambda \in L_1,\end{aligned}$$

with $\nu = \gamma_3\gamma_2(\nu') \in K$ and $\mu = \gamma_3(\mu') \in L_j$. In particular,

$$\alpha = (\gamma_3\gamma_2\gamma_1)^{-1} E^z_{a^{-1}\nu} E^{w_{\epsilon j}}_{b^{-1}\mu} \alpha_M^{\nu_p(a),\nu_p(b)}.$$

Since $\mathrm{spin}(\alpha) = \mathrm{spin}(\alpha_M^{\nu_p(a),\nu_p(b)}) = 1$, we have $\gamma_3\gamma_2\gamma_1 \in \Gamma$. Now we go through the different possibilities for a and b:
If $a = b = 1$, then $E^z_{a^{-1}\nu} E^{w_{\epsilon j}}_{b^{-1}\mu} \in \Gamma_+$ and there is nothing to prove.
Let $a = 1$ and $b = p$. Then $E^z_{a^{-1}\nu} \in \Gamma_+$ and $E^{w_{\epsilon j}}_{p^{-1}\mu} \alpha_M^{0,1}(L) \subset L$ so that we must have $\mathrm{q}(\mu) \in p\mathbb{Z}$ and $(\beta^{-1}(p\lambda), \mu) \in p\mathbb{Z}$ for all $\lambda \in L_1$. Since $\beta^{-1}(pL_1)$ is a rescaled unimodular lattice, this is the case if and only if $\mu \in \beta^{-1}(pL_1) = L_M^{0,1}$.
Let $a = p$ and $b = 1$. Then $E^{w_{\epsilon j}}_{b^{-1}\mu} \in \Gamma_+$ and we have

$$E^z_{p^{-1}\nu} E^{w_{\epsilon j}}_{\mu} = E^{w_{\epsilon j}}_{\mu} E^z_{p^{-1}\nu'}$$

with $\nu' = E^{w_{\epsilon j}}_{-\mu}(\nu)$ and so $E^z_{p^{-1}\nu'} \alpha_M^{1,0}(L) \subset L$. Analogously to the previous case we see that this is the case if and only if $\nu' \in L_M^{1,0}$.
Finally, let $a = b = p$. Again one verifies that $E^z_{p^{-1}\nu} E^{w_{\epsilon j}}_{p^{-1}\mu} \alpha_M^{1,1}(L) \subset L$ if and only if $\nu \in L_M^{1,1}$ and $\mu \in L_M^{0,1}$. This proves that $G(p)$ is in fact a union of the given sets as claimed.

Next we show that the union is disjoint. Let $j, j' \in \{1, \ldots, h\}$, $M \in \{L_j(p) \subset L_1\}$, $M' \in \{L_{j'}(p) \subset L_1\}$ and $a, a', b, b' \in \{1, p\}$. Let

$$\alpha = E^z_{a^{-1}\nu} E^{w_{\epsilon j}}_{b^{-1}\mu} \alpha_M^{\nu_p(a),\nu_p(b)} \in G(p)_M^{\nu_p(a),\nu_p(b)}$$
$$\alpha' = E^z_{a'^{-1}\nu'} E^{w_{\epsilon' j'}}_{b'^{-1}\mu'} \alpha_{M'}^{\nu_p(a'),\nu_p(b')} \in G(p)_{M'}^{\nu_p(a'),\nu_p(b')}$$

for ν, ν', μ, μ' in the appropriate lattices and assume that

$$\alpha' = \gamma\alpha$$

for some $\gamma \in \Gamma$. Then

$$\gamma = \alpha'\alpha^{-1}.$$

We calculate

$$\gamma(z) = \frac{a}{a'} z$$

so that $a = a'$. Next we compute

$$\gamma(w_{\epsilon j}) = \frac{b}{b'} w_{\epsilon' j'} + \left(\frac{(w_{\epsilon' j'}, \nu')}{a'} - \frac{(w_{\epsilon j}, \nu)}{a} \right) z$$

hence, $b = b'$ and $j = j'$. Finally, let $\lambda \in L_j$. Then

$$\gamma(\lambda) = \beta'^{-1}\beta(\lambda) + xz + yw_{\epsilon' j'}$$

for some $x, y \in \mathbb{Z}$ and $\beta'^{-1}\beta \in \mathrm{O}(L_j)$. Then $M' = \beta'(L_j) = \beta(L_j) = M$. This finishes the proof. □

We now want to study how the Hecke operator $T(p)$ commutes with the Φ-operator. First, note that since L is unimodular we have

Proposition 6.2.4. *Let $w, w_1', w_2' \in K$ be primitive isotropic such that $(w, w_1') = (w, w_2') = -1$. Then for any modular form ψ of weight k and trivial character for $\mathrm{SO}(L)^+$ we have*

$$\psi_z|\Phi_{w,w_1'}(\tau) = \psi_z|\Phi_{w,w_2'}(\tau).$$

Proof. We must have $w_2' = w_1' + \lambda + \mathrm{q}(\lambda)w$ for some $\lambda \in K \cap w^\perp \cap w_1'^\perp$, i.e. $w_2' = E^w_\lambda(w_1')$ with $j(E^w_\lambda, itw + \tau w_1') = 1$. Then

$$\begin{aligned}
\psi_z|\Phi_{w,w_2'}(\tau) &= \lim_{t\to+\infty} \psi_z(itw + \tau w_2') \\
&= \lim_{t\to+\infty} \psi_z|_k[E^w_\lambda](itw + \tau w_1') \\
&= \lim_{t\to+\infty} \psi_z(itw + \tau w_1') \\
&= \psi_z|\Phi_{w,w_1'}(\tau).
\end{aligned}$$

□

Since $\Phi_{w,w'}$ is independent of the choice of w', we will from now on simply write Φ_w instead of $\Phi_{w,w'}$.

We investigate how the right coset representatives of $\Gamma\backslash G(p)$ that we have described commute with the Φ-operator.

Lemma 6.2.5. *Let ψ be a modular form of weight k and character χ for Γ, where χ is either trivial or equal to* det. *Let $j \in \{1, \dots, h\}$ and $M \in \{L_j(p) \subset L_1\}$. Then for $\alpha = \alpha_M^{0,0}$ we have*

$$\psi_z|_k[\alpha]|\Phi_w(\tau) = p^{k/2}\psi_z|\Phi_{w_{\epsilon j}}(\tau).$$

If $\alpha = E^{w_{\epsilon j}}_{p^{-1}\mu}\alpha_M^{0,1}$ for some $\mu \in L_M^{0,1}$, then

$$\psi_z|_k[\alpha]|\Phi_w(\tau) = p^{k/2}\psi_z|\Phi_{w_{\epsilon j}}(p\tau).$$

If $\alpha = E^{z}_{p^{-1}\nu}\alpha_M^{1,0}$ for some $\nu \in L_M^{1,0}$, then

$$\psi_z|_k[\alpha]|\Phi_w(\tau) = p^{-k/2}\psi_z|\Phi_{w_{\epsilon j}}\left(\frac{\tau - (\nu, w_{\epsilon j})}{p}\right).$$

If $\alpha = E^{z}_{p^{-1}\nu}E^{w_{\epsilon j}}_{p^{-1}\mu}\alpha_M^{1,1}$ for some $\nu \in L_M^{1,1}$ and $\mu \in L_M^{0,1}$, then

$$\psi_z|_k[\alpha]|\Phi_w(\tau) = p^{-k/2}\psi_z|\Phi_{w_{\epsilon j}}(\tau).$$

Proof. For any $\alpha \in G$ we have

$$\begin{aligned}\psi_z|_k[\alpha]|\Phi_w(\tau) &= \lim_{t\to\infty}\psi_z|_k[\alpha](itw + \tau w')\\ &= \lim_{t\to\infty}p^{k/2}j(\alpha, itw+\tau w')^{-k}\psi_z(\alpha(itw+\tau w')).\end{aligned}$$

We set $Z = itw + \tau w'$ so that $Z_L = Z + z' - it\tau z$.
Let $\alpha = \alpha_M^{0,0}$ or $\alpha = E^{w_{\epsilon j}}_{p^{-1}\mu}\alpha_M^{0,1}$ with $\mu \in L_M^{0,1}$. In both cases we have $\alpha(Z_L) = \alpha(Z) + z' - it\tau p z$. Since $\alpha(Z)$ is orthogonal to z and z' we obtain $j(\alpha, itw+\tau w') = 1$. Hence,

$$\begin{aligned}\psi_z|_k[\alpha]|\Phi_w(\tau) &= \lim_{t\to\infty}p^{k/2}\psi_z(\alpha Z)\\ &= p^{k/2}\lim_{t\to\infty}\psi((\alpha Z)_L)\\ &= p^{k/2}\lim_{t\to\infty}\psi(\alpha(Z_L))\\ &= p^{k/2}\lim_{t\to\infty}\psi(it\alpha(w) + \tau\alpha(w') + z' - it\tau z)\\ &= p^{k/2}\lim_{t\to\infty}\psi_z(it\alpha(w) + \tau\alpha(w'))\end{aligned}$$

since $\alpha(w)$ and $\alpha(w')$ lie in K. If $\alpha = \alpha_M^{0,0}$, then

$$\begin{aligned}\psi_z|_k[\alpha]|\Phi_w(\tau) &= p^{k/2} \lim_{t\to\infty} \psi_z(it\alpha(w) + \tau\alpha(w')) \\ &= p^{k/2} \lim_{t\to\infty} \psi_z(itpw_{\epsilon j} + \tau w_{-\epsilon j}) \\ &= p^{k/2} \psi_z|\Phi_{w_{\epsilon j}}(\tau).\end{aligned}$$

If $\alpha = E_{p^{-1}\mu}^{w_{\epsilon j}} \alpha_M^{0,1}$, then

$$\begin{aligned}\psi_z|_k[\alpha]|\Phi_w(\tau) &= p^{k/2} \lim_{t\to\infty} \psi_z(it\alpha(w) + \tau\alpha(w')) \\ &= p^{k/2} \lim_{t\to\infty} \psi_z(itw_{\epsilon j} + \tau(pw_{-\epsilon j} + p^{-1}\,\mathrm{q}(\mu)w_{\epsilon j} + \mu)) \\ &= p^{k/2} \lim_{t\to\infty} \sum_{\substack{n,m\geq 0 \\ \lambda\in L_j \\ 2nm\geq(\lambda,\lambda)}} a(nw_{\epsilon j} + mw_{-\epsilon j} + \lambda) e^{-2\pi mt} \\ &\qquad e(m\tau p^{-1}\,\mathrm{q}(\mu)) e(np\tau) e(-(\lambda,\mu)) \\ &= p^{k/2} \lim_{t\to\infty} \sum_{n\geq 0} a(nw_{\epsilon j}) e(np\tau) \\ &= p^{k/2} \psi_z|\Phi_{w_{\epsilon j}}(p\tau).\end{aligned}$$

Now let $\alpha = E_{p^{-1}\nu}^{z} \alpha_M^{1,0}$ or $\alpha = E_{p^{-1}\nu}^{z} E_{p^{-1}\mu}^{w_{\epsilon j}} \alpha_M^{1,1}$ for $\nu \in L_M^{1,0}$ or $\nu \in L_M^{1,1}$ respectively and $\mu \in L_M^{0,1}$. Then we have $\alpha(Z_L) = \alpha(Z) + pz' + p^{-1}\,\mathrm{q}(\nu)z + \nu - it\tau z$. Since $\alpha(Z)$ is orthogonal to z, we obtain $(\alpha(Z_L), z) = (pz', z) = -p$. Therefore, $j(\alpha, itw + \tau w') = p$. We thus have

$$\begin{aligned}\psi_z|_k[\alpha]|\Phi_w(\tau) &= \lim_{t\to\infty} p^{-k/2} \psi_z(\alpha(Z)) \\ &= p^{-k/2} \lim_{t\to\infty} \psi((\alpha Z)_L) \\ &= p^{-k/2} \lim_{t\to\infty} \psi(j(\alpha, Z)^{-1} \alpha(Z_L)) \\ &= p^{k/2} \lim_{t\to\infty} \psi(it\alpha(w) + \tau\alpha(w') + pz' + p^{-1}\,\mathrm{q}(\nu)z + \nu - it\tau z) \\ &= p^{-k/2} \lim_{t\to\infty} \psi\left(\frac{it}{p}\alpha(w) + \frac{\tau}{p}\alpha(w') + \frac{\nu}{p} + z' + \frac{\mathrm{q}(\nu) - itp\tau}{p^2} z\right) \\ &= p^{-k/2} \lim_{t\to\infty} \psi_z\left(\frac{it}{p}\alpha(w)|_K + \frac{\tau}{p}\alpha(w')|_K + \frac{\nu}{p}\right),\end{aligned}$$

where $\alpha(w)|_K$ and $\alpha(w')|_K$ denotes the projection of $\alpha(w)$ respectively $\alpha(w')$ to K (Note that $\alpha(w)$ and $\alpha(w')$ are orthogonal to z but not necessarily orthogonal to

$z')$. If $\alpha = E^z_{p^{-1}\nu}\alpha^{1,0}_M$, then

$$\begin{aligned}
\psi_z|_k[\alpha]|\Phi_w(\tau) &= p^{-k/2}\lim_{t\to\infty}\psi_z\left(itw_{\epsilon j} + \frac{\tau}{p}w_{-\epsilon j} + \frac{\nu}{p}\right) \\
&= p^{-k/2}\lim_{t\to\infty}\sum_{\substack{n,m\geq 0\\ \lambda\in L_j\\ 2nm\geq(\lambda,\lambda)}} a(nw_{\epsilon j} + mw_{-\epsilon j} + \lambda)e^{-2\pi mt} \\
&\qquad e(n\tau/p)e(-n(\nu, w_{\epsilon j})/p)e(-(\lambda,\nu)/p) \\
&= p^{-k/2}\sum_{n\geq 0} a(nw_{\epsilon j})e\left(n\frac{\tau-(\nu,w_{\epsilon j})}{p}\right) \\
&= p^{-k/2}\psi_z|\Phi_{w_{\epsilon j}}\left(\frac{\tau-(\nu,w_{\epsilon j})}{p}\right).
\end{aligned}$$

Finally, if $\alpha = E^z_{p^{-1}\nu}E^{w_{\epsilon j}}_{p^{-1}\mu}\alpha^{1,1}_M$, then

$$\begin{aligned}
\psi_z|_k[\alpha]|\Phi_w(\tau) &= p^{-k/2}\lim_{t\to\infty}\psi_z\left(\frac{it}{p}w_{\epsilon j} + \tau\left(w_{-\epsilon j} + \frac{\mu}{p}\right) + \frac{\nu}{p}\right) \\
&= p^{-k/2}\sum_{n\geq 0} a(nw_{\epsilon j})e\left(n\left(\tau - \frac{(\nu, w_{\epsilon j})}{p}\right)\right).
\end{aligned}$$

Note that we have $(w_{\epsilon j}, \nu) \in p\mathbb{Z}$ and so

$$\begin{aligned}
\psi_z|_k[\alpha]|\Phi_w(\tau) &= p^{-k/2}\sum_{n\geq 0} a(nw_{\epsilon j})e(n\tau) \\
&= p^{-k/2}\psi_z|\Phi_{w_{\epsilon j}}(\tau).
\end{aligned}$$

□

We will see in the next theorem that applying the Φ-operator after the Hecke operator $\mathcal{T}(p)$ is the same as first applying a linear combination of Φ-operators and then a classical Hecke operator on the space of modular forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$. For this purpose we consider the Hecke algebra $\mathscr{H}(\mathrm{SL}_2(\mathbb{Z}), \mathrm{GL}_2(\mathbb{Q})_+)$. By the elementary divisor theorem for the symplectic group (cf. Lemma 4.1.1) any double coset of integer matrices in $\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{GL}_2(\mathbb{Q})_+/\mathrm{SL}_2(\mathbb{Z})$ has a representative of the form

$$\begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix}$$

for positive integers $l_1 \mid l_2$ and it is well-known that

$$\mathrm{SL}_2(\mathbb{Z})\begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix}\mathrm{SL}_2(\mathbb{Z}) = \bigcup_{\substack{ad=l_1l_2\\ b\in\mathbb{Z}/d\mathbb{Z}\\ \gcd(a,d,b)=l_1}} \mathrm{SL}_2(\mathbb{Z})\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

In particular,

$$\mathrm{SL}_2(\mathbb{Z})\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\mathrm{SL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix} \cup \bigcup_{b=0}^{p-1}\mathrm{SL}_2(\mathbb{Z})\begin{pmatrix}1 & b\\ 0 & p\end{pmatrix}.$$

We denote the space of modular forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$ by M_k. The Hecke algebra $\mathscr{H}(\mathrm{SL}_2(\mathbb{Z}), \mathrm{GL}_2(\mathbb{Q})_+)$ acts on M_k by

$$f|_k \mathrm{SL}_2(\mathbb{Z})\delta\mathrm{SL}_2(\mathbb{Z}) = \sum_{\delta_i \in \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z})\delta\mathrm{SL}_2(\mathbb{Z})} f|_k[\delta_i]$$

and we define the Hecke operator

$$T(l) := l^{k/2-1}\cdot \mathrm{SL}_2(\mathbb{Z})\begin{pmatrix}1 & 0\\ 0 & l\end{pmatrix}\mathrm{SL}_2(\mathbb{Z}) \in \mathscr{H}(\mathrm{SL}_2(\mathbb{Z}), \mathrm{GL}_2(\mathbb{Q})_+) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Note that the operator $T(l^2)$ coincides with the operator by the same name acting on the space of vector-valued modular forms when the discriminant form is trivial defined in Section 4.1. (For more details on classical Hecke operator cf. for example [21].)

Recall that $L_1 \dots, L_h \in II_{n-2,2}$ is a complete system of representatives and $w_{\pm j} \in K$ are primitive isotropic vectors with $(w_{+j}, w_{-j}) = -1$ and $L_j = K \cap w_{+j}^{\perp} \cap w_{-j}^{\perp}$, where $K = L \cap z^{\perp} \cap z'^{\perp}$ for primitive isotropic vectors $z, z' \in L$ with $(z, z') = -1$. We furthermore set $w = w_{+1}$ and $w' = w_{-1}$.

We define representation numbers $r_j^{\chi}(p)$ in the following way: If χ is trivial, then

$$r_j^{\chi}(p) := \#\{L_j(p) \subset L_1\}.$$

If $\chi = \det$, we do the following: Recall that for any $M \subset \{L_j(p) \subset L_1\}$ we chose a $\beta : L_j \to M$ and defined $\epsilon = \pm 1$ such that $\alpha_M^{0,0}$ has positive determinant, which in general depends on the choice of β. If $\alpha = \alpha_M^{0,0}$ and α' is the corresponding element for a different choice $\beta' : L_j \to M$ and ϵ' accordingly, then $\alpha'\alpha^{-1} \in \Gamma_+$ is given by

$$\begin{aligned} z &\mapsto z\\ z' &\mapsto z'\\ w_{\epsilon j} &\mapsto w_{\epsilon' j}\\ w_{-\epsilon j} &\mapsto w_{-\epsilon' j}\\ \lambda &\mapsto \beta'^{-1}\beta(\lambda) \quad \text{for } \lambda \in L_j, \end{aligned}$$

where $\beta'^{-1}\beta \in \mathrm{O}(L_j)$ with $\det(\beta'^{-1}\beta) = \epsilon\epsilon'$. So if L_j is chiral, then $\epsilon = \epsilon'$ does not depend on the choice of β, but only on M. In this case we set

$$r_j^\chi(p) := \#\{M \in \{L_j(p) \subset L_1\} \mid \epsilon = +1\} - \#\{M \in \{L_j(p) \subset L_1\} \mid \epsilon = -1\}.$$

If L_j is achiral, we set $r_j^\chi(p) := 0$. Recall that for $j = 1$ we have $\epsilon p^{n/2-1} = \det(\beta)$. We obtain

Theorem 6.2.6. *Let χ be either trivial or equal to* det. *Then the diagram*

$$\begin{array}{ccc} \mathcal{M}_k(\Gamma,\chi) & \xrightarrow{\sum_{j=1}^h r_j^\chi(p)\Phi_{w_j}} & \mathrm{M}_k \\ {\scriptstyle \mathcal{T}(p)}\big\downarrow & & \big\downarrow{\scriptstyle p^{n/2-k/2}T(p)+p^{n-k/2-1}+p^{k/2}} \\ \mathcal{M}_k(\Gamma,\chi) & \xrightarrow{\Phi_w} & \mathrm{M}_k \end{array}$$

commutes.

Proof. For any $j = 1, \ldots, h$ and $M \in \{L_j(p) \subset L_1\}$ let $S_M \subset L_M^{0,1}$ be a full system of representatives for $L_M^{0,1}/pL_j$. Then $\{\nu' + bw_{-\epsilon j} \mid \nu' \in S_M,\ b \in \{0, \ldots, p-1\}\}$ is a full system of representatives for $L_M^{1,0}/pK$ and by Lemma 6.2.2

$$\{\nu' + bw_{\epsilon j} \mid \nu' \in S_M,\ b \in \{0, \ldots, p-1\}\} \times S_M$$

is a full system of representatives for $L_M^{1,1} \times L_M^{0,1}/\sim_M$. Note that for $\nu = \nu' + bw_{-\epsilon j}$ with $\nu' \in S_M$ we have

$$\frac{\tau - (\nu, w_{\epsilon j})}{p} = \frac{\tau + b}{p}.$$

Now by Lemma 6.2.1 and Proposition 6.2.3 we have

$$\begin{aligned} \psi_z|_k\mathcal{T}(p) = \sum_{j=1}^h \sum_{M\in\{L_j(p)\subset L_1\}} \Bigg(& \psi_z|_k[\alpha_M^{0,0}] + \sum_{\mu\in S_M} \psi_z|_k[E_{p^{-1}\mu}^{w_{\epsilon j}}\alpha_M^{0,1}] \\ & + \sum_{\nu'\in S_M} \sum_{b=0}^{p-1} \psi_z|_k[E_{p^{-1}(\nu'+bw_{-\epsilon j})}^z \alpha_M^{1,0}] \\ & + \sum_{\nu'\in S_M} \sum_{b=0}^{p-1} \sum_{\mu\in S_M} \psi_z|_k[E_{p^{-1}(\nu'+bw_{\epsilon j})}^z E_{p^{-1}\mu}^{w_{\epsilon j}} \alpha_M^{1,0}] \Bigg). \end{aligned}$$

Employing Lemma 6.2.5 yields

$$\begin{aligned}
&\psi_z|_k\mathcal{T}(p)|\Phi_w(\tau)\\
&\quad=\sum_{j=1}^{h}\sum_{M\in\{L_j(p)\subset L_1\}}\Bigg(p^{k/2}\psi_z|\Phi_{w_\epsilon j}(\tau)+p^{k/2}\sum_{\mu\in S_M}\psi_z|\Phi_{w_{\epsilon j}}(p\tau)\\
&\qquad+p^{-k/2}\sum_{\nu'\in S_M}\sum_{b=0}^{p-1}\psi_z|\Phi_{w_{\epsilon j}}\left(\frac{\tau+b}{p}\right)\\
&\qquad+p^{-k/2}\sum_{\nu'\in S_M}\sum_{b=0}^{p-1}\sum_{\mu\in S_M}\psi_z|\Phi_{w_{\epsilon j}}(\tau)\Bigg)\\
&\quad=\sum_{j=1}^{h}\sum_{M\in\{L_j(p)\subset L_1\}}\Bigg(p^{n/2-k/2}\left(p^{k-1}\psi_z|\Phi_{w_{\epsilon j}}(p\tau)+\sum_{b=0}^{p-1}p^{-1}\psi_z|\Phi_{w_{\epsilon j}}\left(\frac{\tau+b}{p}\right)\right)\\
&\qquad+p^{n-k/2-1}\psi_z|\Phi_{w_{\epsilon j}}(\tau)+p^{k/2}\psi_z|\Phi_{w_{\epsilon j}}(\tau)\Bigg)\\
&\quad=\sum_{j=1}^{h}\sum_{M\in\{L_j(p)\subset L_1\}}\Bigg(p^{n/2-k/2}\psi_z|\Phi_{w_{\epsilon j}}|_kT(p)(\tau)\\
&\qquad+p^{n-k/2-1}\psi_z|\Phi_{w_{\epsilon j}}(\tau)+p^{k/2}\psi_z|\Phi_{w_{\epsilon j}}(\tau)\Bigg).
\end{aligned}$$

Finally, note that $\psi_z|\Phi_{w_{\epsilon j}}=\psi_z|\Phi_{w_{+j}}$ if χ is trivial and

$$\psi_z|\Phi_{w_{\epsilon j}}=\begin{cases}\epsilon\psi_z|\Phi_{w_{+j}} & \text{if } L_j \text{ is chiral}\\ 0 & \text{else}\end{cases}$$

if $\chi=\det$. Therefore,

$$\begin{aligned}
&\psi_z|_k\mathcal{T}(p)|\Phi_w(\tau)\\
&\quad=\sum_{j=1}^{h}r_j^{\chi}(p)\left(p^{n/2-k/2}\psi_z|\Phi_{w_j}|_kT(p)(\tau)+p^{n-k/2-1}\psi_z|\Phi_{w_j}(\tau)+p^{k/2}\psi_z|\Phi_{w_j}(\tau)\right).
\end{aligned}$$

□

Chapter 7

Eigenvalues of Borcherds' Φ_{12}

An important examples of an automorphic product is Borcherds' Φ_{12}. It is the multiplicative Borcherds lift of $1/\Delta$ on the lattice $L = II_{26,2}$. We will recall some properties of Φ_{12}. It is an eigenform for the Hecke algebra $\mathscr{H}(\mathrm{O}(L)^+, \mathrm{GO}(L_{\mathbb{Q}})^+_+)$ studied in the previous section. We will compute an explicit formula for the eigenvalue $\lambda(p)$ of Φ_{12} corresponding to the Hecke operator $\mathcal{T}(p)$ using the Satake isomorphism.

This chapter is based on joint work with Moritz Dittmann and Nils Scheithauer (cf. [22]).

7.1 Definition of Φ_{12} and its eigenvalues

Let $L = II_{26,2}$ and let Φ_{12} be the multiplicative Borcherds lift of $1/\Delta$ on L, where $\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$ is the modular discriminant. Then Φ_{12} is a holomorphic automorphic form of singular weight 12 and character det for the group $\mathrm{O}(L)^+$ (cf. [6, §10]).

For any primitive isotropic $z, z' \in L$ with $(z, z') = -1$ we write $L = II_{25,1} \oplus \langle z, z'\rangle$. Let $w, w' \in II_{25,1}$ be primitive isotropic with $(w, w') = -1$ such that $w + w' \in \mathcal{C}$, where $\mathcal{C}$ is as in the last two chapters (see page 147). Then we have $II_{25,1} = N \oplus \langle w, w'\rangle$, where N is a Niemeier lattice. In fact, the orbits of primitive isotropic vectors in $II_{25,1}$ under $\mathrm{O}(II_{25,1})$ are in 1-to-1-correspondence with the Niemeier lattices. Now let $\rho \in II_{25,1}$ be a primitive isotropic vector corresponding to the Leech lattice Λ, i.e. $II_{25,1} = \Lambda \oplus \langle \rho, \rho'\rangle$ for some isotropic $\rho' \in II_{25,1}$ with $(\rho, \rho') = -1$ and we again assume that $\rho + \rho' \in \mathcal{C}$. Then Φ_{12} has a product expansion on $\mathcal{H}_{z,z'}$ given by

$$\Phi_{12}(Z) = e(-(\rho, Z)) \prod_{\alpha \in II_{25,1}^+} (1 - e(-(\alpha, Z)))^{c(-(\alpha,\alpha)/2)}, \tag{7.1.1}$$

where $II_{25,1}^{+}$ denotes the positive roots of $II_{25,1}$ and $1/\Delta(\tau) = \sum_{n=-1}^{\infty} c(n)q^n$. The Weyl group W is generated by the reflections in elements of $II_{25,1}$ of norm 2. Since Φ_{12} is antisymmetric under this group and of singular weight, it follows that

$$\Phi_{12}(Z) = \sum_{w \in W} \det(w) e(-(w\rho, Z)) \prod_{n=1}^{\infty} (1 - e(-(nw\rho, Z)))^{24}. \qquad (7.1.2)$$

Recall that $II_{25,1} = N \oplus \langle w, w' \rangle$ for primitive isotropic vectors $w, w' \in II_{25,1}$. Note that the Leech lattice is the only Niemeier lattice which is chiral. By Lemma 5.2.4, Φ_{12} should be identically 0 on those 1-dimensional boundary components corresponding to the Niemeier lattices other than the Leech lattice. Indeed, it follows from equation (7.1.2) that

$$\Phi_{12}|\Phi_w(\tau) = \begin{cases} \det(\gamma)\Delta(\tau) & \text{if } \gamma(\rho) = w \text{ for some } \gamma \in \mathrm{O}(II_{25,1})^+ \\ 0 & \text{if } N \not\cong \Lambda \end{cases}$$

and note that there exists a $\gamma \in \mathrm{O}(II_{25,1})^+$ with $\gamma(\rho) = w$ if and only if $N \cong \Lambda$.

Interestingly, Φ_{12} is the only holomorphic automorphic product of singular weight on a unimodular lattice (see [62] and [23]). In particular, $\dim(\mathcal{M}_{12}(\mathrm{O}(L)^+, \det)) = 1$ (see also [6, §10]) so that we must have

$$\Phi_{12}|_{12}\mathcal{T}(p) = \lambda(p)\Phi_{12}$$

for some $\lambda(p) \in \mathbb{C}$.

Let K, $w_{\pm 1}, \ldots, w_{\pm h}$ and $L_1, \ldots, L_h$ be as in the previous chapter. Then $K = II_{25,1}$, $h = 24$ and $L_1, \ldots, L_{24} \in II_{24,0}$ are the 24 Niemeier lattices. Let $L_1 = \Lambda$ be the Leech lattice and $w_1 = \rho$ be the same as in (7.1.2). Then

$$\begin{aligned} r(p) &:= r_1^{\det}(p) \\ &= \#\{M \subset \Lambda \mid \text{there exists a } \beta : \Lambda(p) \xrightarrow{\sim} M \text{ such that } \det(\beta) = +1\} \\ &\qquad - \#\{M \subset \Lambda \mid \text{there exists a } \beta : \Lambda(p) \xrightarrow{\sim} M \text{ such that } \det(\beta) = -1\} \end{aligned}$$

and $r_j^{\det}(p) = 0$ for $j > 1$ since Λ is the only Niemeier lattice which is chiral. From Theorem 6.2.6 now follows

Proposition 7.1.1. *The eigenvalue $\lambda(p)$ of Φ_{12} corresponding to $\mathcal{T}(p)$ is given by*

$$\lambda(p) = r(p)(p^7\tau(p) + p^{19} + p^6).$$

Proof. By Theorem 6.2.6 we have

$$\begin{aligned}\lambda(p)\Phi_{12}|\Phi_\rho &= \Phi_{12}|_{12}\mathcal{T}(p)|\Phi_\rho \\ &= r(p)(p^7\Phi_{12}|\Phi_\rho|_{12}T(p) + p^{19}\Phi_{12}|\Phi_\rho + p^6\Phi_{12}|\Phi_\rho) \\ &= r(p)(p^7\tau(p) + p^{19} + p^6)\Phi_{12}|\Phi_\rho,\end{aligned}$$

where we have used that $\Phi_{12}|\Phi_\rho = \Delta$ and $\Delta|_{12}T(p) = \tau(p)\Delta$. □

Finally, we want to derive a formula for $r(p)$. We choose an embedding $\Lambda \subset \mathbb{R}^{24}$ such that the bilinear form $(\cdot,\cdot)$ on Λ extends to the standard scalar product on $\mathbb{R}^{24}$. Consider the Siegel theta series

$$\theta(Z) := \theta^{(24)}_{\Lambda,\det} = \sum_{\underline{\lambda}\in\Lambda^{24}} \det(\underline{\lambda}) e^{\pi i \operatorname{tr}((\underline{\lambda},\underline{\lambda})Z)},$$

where $\det(\underline{\lambda})$ is the determinant of the 24×24-matrix $\underline{\lambda} = (\lambda_1,\ldots,\lambda_{24})$ with $\lambda_i \in \Lambda \subset \mathbb{R}^{24}$ as a column vector. Then θ is a cusp form of weight 13 for the group $\mathrm{Sp}_{48}(\mathbb{Z})$. The Fourier expansion of θ is given by

$$\theta(Z) = \sum_{\substack{S=S^T \text{ even} \\ S\geq 0}} a(S) e^{\pi i \operatorname{tr}(SZ)}$$

and if G is any Gram matrix of Λ, then $a(pG) = p^{12}r(p)a(G)$. Note that $a(G) = \pm\#\operatorname{Aut}(\Lambda)$.

Consider the Hecke algebra

$$\mathscr{H}^{(n)} := \mathscr{H}(\mathrm{Sp}_{2n}(\mathbb{Z}), \mathrm{GSp}_{2n}(\mathbb{Q})_+),$$

where $\mathrm{GSp}_{2n}(\mathbb{Q})_+ = \{M \in \mathrm{GL}_{2n}(\mathbb{Q}) \mid M^T J_n M = lJ_n \text{ for some } l \in \mathbb{Q}_{>0}\}$ denotes the group of symplectic similitude matrices of rank $2n$. Analogous to the Hecke algebras we have already seen, it acts on the space of Siegel modular forms of weight k for $\mathrm{Sp}_{2n}(\mathbb{Z})$ by

$$f|_k T = \sum_{M\in \mathrm{Sp}_{2n}(\mathbb{Z})\backslash T} f|_k[M],$$

where $T = \mathrm{Sp}_{2n}(\mathbb{Z})M\mathrm{Sp}_{2n}(\mathbb{Z}) \in \mathscr{H}^{(n)}$ is some double coset. In particular, for a prime p define

$$T^{(n)}(p) := p^{\frac{nk-n(n+1)}{2}} \cdot \mathrm{Sp}_{2n}(\mathbb{Z}) \begin{pmatrix} I_n & 0 \\ 0 & pI_n \end{pmatrix} \mathrm{Sp}_{2n}(\mathbb{Z}) \in \mathscr{H}^{(n)} \otimes_{\mathbb{Z}} \mathbb{C}.$$

The following is Theorems 5.2 and 5.3 in [29] and is shown using an explicit formula for the action of a Hecke operator on theta series.

Theorem 7.1.2. *The theta series θ is a simultaneous eigenform form for the Hecke algebra $\mathscr{H}^{(24)} \otimes_{\mathbb{Z}} \mathbb{C}$. Let p be a prime. The eigenvalue $\mu(p)$ of θ corresponding to the Hecke operator $T^{(24)}(p)$ is given by*

$$\mu(p) = \frac{a(pG)}{a(G)} \cdot \prod_{j=1}^{12} (1 + p^{-j}).$$

In particular, we find that

$$r(p) = \mu(p) \cdot p^{-12} \prod_{j=1}^{12} (1 + p^{-j})^{-1}.$$

Note that in [31] the slash operator of an element $M \in \mathrm{GSp}_{2n}(\mathbb{Q})_+$ does not contain the factor $\det(M)^{k/2}$ and the operators $T^{(24)}(p)$ are not normalized as they are in the present work. Therefore, the formula in [31] slightly differs from the one given here.

7.2 The Satake isomorphism

In [18] and [19] the Satake parameters of θ, which encode its eigenvalues, where determined using deep results from the theory of automorphic representations. We will explain what Satake parameters are and how they can be used to calculate the eigenvalues of θ. We will not explain all the terminology or give all details, as far as we can use them as a black box. We remark that it should also be possible to work directly with the Hecke algebra of the orthogonal group O_{24} acting on the Niemeier latices to compute $r(p)$, but we do not do this here.

Local Hecke algebras

Let $\mathbb{G}_{\mathrm{m}}$ be the $\mathbb{Z}$-group scheme given by $A \mapsto A^\times$ on the category of rings. We furthermore define the $\mathbb{Z}$-group schemes given by

$$\begin{aligned} \mathrm{GSp}_{2n}(A) &\coloneqq \{M \in \mathrm{Mat}_{2n}(A) \mid M^T J_n M = l(M) J_n \text{ for some } l(M) \in A^\times\}, \\ \mathrm{Sp}_{2n} &\coloneqq \ker(l), \\ \mathrm{PGSp}_{2n} &\coloneqq \mathrm{GSp}_{2n}/\mathbb{G}_{\mathrm{m}} I_{2n}, \end{aligned}$$

where $l : \mathrm{GSp}_{2n} \to \mathbb{G}_{\mathrm{m}}$ is called the similitude factor and the quotient is formal as fppf-sheaves. Note that $\mathrm{PGSp}_{2n}(A) = \mathrm{GSp}_{2n}(A)/A^\times \cdot I_{2n}$ for all local rings A (by Hilbert's theorem 90). When A is a subring of $\mathbb{R}$ let us also set

$$\mathrm{GSp}_{2n}(A)_+ = \{M \in \mathrm{GSp}_{2n}(A) \mid l(M) > 0\}$$

and $\mathrm{PGSp}_{2n}(A)_+ = \mathrm{GSp}_{2n}(A)_+/A^\times I_{2n}$.

Similar to the situation of the orthogonal groups, the operator $T^{(n)}(p)$ is also in the local Hecke algebra $\mathscr{H}\left(\mathrm{Sp}_{2n}(\mathbb{Z}), \mathrm{GSp}_{2n}(\mathbb{Z}[\frac{1}{p}])_+\right) \otimes_{\mathbb{Z}} \mathbb{C}$. Since diagonal matrices act trivially on Siegel modular forms, $T^{(n)}(p)$ acts identically to its projection onto $\mathscr{H}\left(\mathrm{PGSp}_{2n}(\mathbb{Z})_+, \mathrm{PGSp}_{2n}(\mathbb{Z}[\frac{1}{p}])_+\right) \otimes_{\mathbb{Z}} \mathbb{C}$. From the elementary divisor theorem for the symplectic group it follows that the natural map

$$\mathrm{PGSp}_{2n}(\mathbb{Z})_+ \backslash \mathrm{PGSp}_{2n}\left(\mathbb{Z}\left[\tfrac{1}{p}\right]\right)_+ / \mathrm{PGSp}_{2n}(\mathbb{Z})_+ \\ \to \mathrm{PGSp}_{2n}(\mathbb{Z}_p) \backslash \mathrm{PGSp}_{2n}(\mathbb{Q}_p) / \mathrm{PGSp}_{2n}(\mathbb{Z}_p)$$

is a bijection (cf. [18, Section 4.5.5]). We can therefore consider $T^{(n)}(p)$ as an element of $\mathscr{H}(\mathrm{PGSp}_{2n}(\mathbb{Z}_p), \mathrm{PGSp}_{2n}(\mathbb{Q}_p)) \otimes_{\mathbb{Z}} \mathbb{C}$. Furthermore, it is not difficult to see from the elementary divisor theorem that

$$\mathrm{Sp}_{2n}(\mathbb{Z}_p) \backslash \mathrm{Sp}_{2n}(\mathbb{Q}_p) / \mathrm{Sp}_{2n}(\mathbb{Z}_p) \hookrightarrow \mathrm{PGSp}_{2n}(\mathbb{Z}_p) \backslash \mathrm{PGSp}_{2n}(\mathbb{Q}_p) / \mathrm{PGSp}_{2n}(\mathbb{Z}_p)$$

and this implies

$$\mathscr{H}(\mathrm{Sp}_{2n}(\mathbb{Z}_p), \mathrm{Sp}_{2n}(\mathbb{Q}_p)) \hookrightarrow \mathscr{H}(\mathrm{PGSp}_{2n}(\mathbb{Z}_p), \mathrm{PGSp}_{2n}(\mathbb{Q}_p)).$$

The Langlands dual group

We will now describe the *Langlands dual group* as introduced by R. Langlands. We use as reference [18, Section 6.1] and [37].

Let X and $X^\vee$ be free abelian groups of finite rank endowed with a perfect pairing $\langle \cdot, \cdot \rangle : X \times X^\vee \to \mathbb{Z}$. Let $\Phi \subset X$ and $\Phi^\vee \subset X^\vee$ be finite subsets endowed with a bijection denoted by $\alpha \mapsto \alpha^\vee$. Assume that for any $\alpha \in \Phi$ we have $\langle \alpha, \alpha^\vee \rangle = 2$ and $\sigma_\alpha(\Phi) = \Phi$ and $\sigma_{\alpha^\vee}(\Phi^\vee) = \Phi^\vee$, where

$$\sigma_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha \quad \text{and} \quad \sigma_{\alpha^\vee}(x) = x - \langle x, \alpha \rangle \alpha^\vee.$$

Then

$$(X, \Phi, X^\vee, \Phi^\vee)$$

is called a *root datum*. The elements in Φ (resp. in $\Phi^\vee$) are called roots (resp. coroots). Furthermore,

$$(X, \Phi, X^\vee, \Phi^\vee)^\vee := (X^\vee, \Phi^\vee, X, \Phi)$$

is also a root datum called the *dual root datum* of $(X, \Phi, X^\vee, \Phi^\vee)$.

Let k be an algebraically closed field and let G be a connected split reductive k-group. We fix a maximal torus T of G. We denote by

$$X^*(T) := \mathrm{Hom}(T, \mathbb{G}_\mathrm{m}) \quad \text{and} \quad X_*(T) := \mathrm{Hom}(\mathbb{G}_\mathrm{m}, T)$$

the free abelian groups of finite rank consisting of the characters and the cocharacters of T, respectively. They are naturally endowed with a perfect pairing $\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \to \mathrm{Hom}(\mathbb{G}_\mathrm{m}, \mathbb{G}_\mathrm{m}) \cong \mathbb{Z}$ (Note that any $f \in \mathrm{Hom}(\mathbb{G}_\mathrm{m}, \mathbb{G}_\mathrm{m})$ is of the form $f(z) = z^n$ for some $n \in \mathbb{Z}$). Let $\Phi := \Phi(G, T) \subset X^*(T)$ be the finite set of roots of G, i.e. the characters of T occurring in the adjoint representation of $\mathrm{Lie}(G)$ and let $\Phi^\vee := \Phi^\vee(G, T) \subset X_*(T)$ be the finite set of coroots (see [18, Section 6.1.1]). Then

$$\Psi(G) = (X^*(T), \Phi, X_*(T), \Phi^\vee)$$

is a root datum and one can show that it is up to isomorphism independent of the choice of T. Now a *Langlands dual group* of G consists of a reductive $\mathbb{C}$-group $\widehat{G}$ and an isomorphism $\Psi(\widehat{G}) \xrightarrow{\sim} \Psi(G)^\vee$. The $\mathbb{C}$-group $\widehat{G}$ is then uniquely determined by G, up to inner isomorphisms, i.e. isomorphisms of the form $x \mapsto gxg^{-1}$ for some $g \in G(\mathbb{C})$. Note that if we consider based root data, then there exists exactly one pinned group $\widehat{G}$ such that the based root datum of $\widehat{G}$ is equal to the dual based root datum of G. Therefore, it makes sense to speak of *the* Langlands dual group.

We consider three examples (cf. [18, Section 6.1.3]). Let $\widetilde{G} = \mathrm{GSp}_{2n}$ with maximal torus

$$\widetilde{T}(A) = \{\mathrm{diag}(z_1, \ldots, z_n, z_0 z_n^{-1}, \ldots, z_0 z_1^{-1}) \in \mathrm{GSp}_{2n}(A) \mid z_0, z_1, \ldots, z_n \in A^\times\}.$$

For $t = \mathrm{diag}(z_1, \ldots, z_n, z_0 z_n^{-1}, \ldots, z_0 z_1^{-1}) \in \widetilde{T}(A)$ let $\epsilon_i \in X^*(\widetilde{T})$ be the character $\epsilon_i(t) = z_i$. Then $(\epsilon_0, \epsilon_1, \ldots, \epsilon_n)$ is a basis of $X^*(T) \cong \mathbb{Z}^{n+1}$ and $\Phi(\widetilde{G}, \widetilde{T})$ consists of the elements $\pm(\epsilon_i - \epsilon_j)$ for $1 \leq i < j \leq n$ and $\pm(\epsilon_i + \epsilon_j - \epsilon_0)$ for $1 \leq i \leq j \leq n$. Let $(\epsilon_0^*, \epsilon_1^*, \ldots, \epsilon_n^*)$ be the dual basis of $(\epsilon_0, \epsilon_1, \ldots, \epsilon_n)$. Then $(\epsilon_i - \epsilon_j)^\vee = \epsilon_i^* - \epsilon_j^*$ and $(\epsilon_i + \epsilon_j - \epsilon_0)^\vee = \epsilon_i^* + \epsilon_j^*$ for $i < j$ and $(2\epsilon_i - \epsilon_0)^\vee = \epsilon_i^*$. The Langlands dual group of GSp_{2n} is

$$\widehat{\mathrm{GSp}_{2n}} \cong \mathrm{GSpin}_{2n+1},$$

the general spin group of rank $2n + 1$.

Now let $G = \mathrm{Sp}_{2n}$ with maximal torus $T = \widetilde{T} \cap G$. Then the character ϵ_0 acts trivially on T and we get $X^*(T) = X^*(\widetilde{T})/\mathbb{Z}\epsilon_0 \cong \mathbb{Z}^n$. The group of cocharacters is given by $X_*(T) = \epsilon_0^{*\perp} \subset X_*(\widetilde{T})$. We have

$$\widehat{\mathrm{Sp}_{2n}} \cong \mathrm{SO}_{2n+1}.$$

Finally, consider $G = \mathrm{PGSp}_{2n}$ with maximal torus $\mathrm{P}\widetilde{T} = \widetilde{T}/\mathbb{G}_\mathrm{m} I_{2n}$. The central cocharacter $\zeta := 2\epsilon_0^* + \sum_{i=1}^n \epsilon_i^*$ acts trivially modulo $\mathbb{G}_\mathrm{m} I_{2n}$ and so $X^*(\mathrm{P}\widetilde{T}) = \zeta^\perp \cong \mathbb{Z}^n$ and $X_*(\mathrm{P}\widetilde{T}) = X_*(\widetilde{T})/\mathbb{Z}\zeta$. The Langlands dual group of PGSp_{2n} is

$$\widehat{\mathrm{PGSp}_{2n}} \cong \mathrm{Spin}_{2n+1},$$

the spin group of rank $2n+1$. We will later describe the spin group, which is a double cover of SO_{2n+1}, in more detail.

The Satake isomorphism

Let G be a connected split reductive $\mathbb{Z}$-group. We define the Hecke ring $\mathscr{H}_p(G) := \mathscr{H}(G(\mathbb{Z}_p), G(\mathbb{Q}_p))$. Let $\widehat{G}$ be a Langlands dual group of G. The isomorphism classes of finite dimensional $\mathbb{C}$-representations of $\widehat{G}$ together with direct sums form an abelian semigroup. Its associated group of fractions $\mathrm{Rep}(\widehat{G})$ consists of formal differences $[\rho_1] - [\rho_2]$ modulo the equivalence relation generated by $[\rho_1] - [\rho_2] \sim [\rho_1 \oplus \rho] - [\rho_2 \oplus \rho]$ for finite dimensional $\mathbb{C}$-representations ρ of $\widehat{G}$. Let ρ_0 be the trivial irreducible representation. Then $[\rho] \mapsto [\rho] - [\rho_0]$ for ρ some irreducible $\mathbb{C}$-representation defines a homomorphism from the free group generated by the isomorphism classes of irreducible, finite dimensional $\mathbb{C}$-representations of $\widehat{G}$ to $\mathrm{Rep}(\widehat{G})$. Since any finite dimensional representation of $\widehat{G}$ is isomorphic to a unique sum of irreducible representations, this is a bijection and we identify the two groups. The tensor product $(\rho, \sigma) \mapsto \rho \otimes \sigma$ defines a commutative ring structure on $\mathrm{Rep}(\widehat{G})$. An element $\sum_i a_i[\rho_i] \in \mathrm{Rep}(\widehat{G})$, where $a_i \in \mathbb{Z}$ and the $[\rho_i]$ are isomorphism classes of irreducible $\mathbb{C}$-representations of $\widehat{G}$, is called a *virtual representations*.

There exists a canonical ring isomorphism

$$\mathrm{Sat} : \mathscr{H}_p(G) \otimes \mathbb{C} \xrightarrow{\sim} \mathrm{Rep}(\widehat{G}) \otimes \mathbb{C}$$

called the *Satake isomorphism* introduced by Satake [58] and revisited by Langlands [45, Section 2]. We will not explicitly describe the Satake isomorphism and refer to [58] and [37] for details. The idea is the following: By elementary divisor theorems similar to [30, Hilfssatz IV.1.12] and Theorem 6.1.12, $\mathscr{H}_p(G)$ has a basis consisting of characteristic functions of $G(\mathbb{Z}_p)\lambda(p)G(\mathbb{Z}_p)$, where λ is a cocharacter of a maximal torus $T \subset G$. By the theorem of the highest weight, the irreducible representations of $\widehat{G}$, which form a basis of $\mathrm{Rep}(\widehat{G})$, are in correspondence to certain characters of a maximal torus $\widehat{T} \subset \widehat{G}$. Because the cocharacters of T are just the characters of $\widehat{T}$ this gives a natural isomorphism between the two rings. Note however, that in general $G(\mathbb{Z}_p)\lambda(p)G(\mathbb{Z}_p)$ is not necessarily mapped to the representation corresponding to λ.

Suppose that M is a $\mathbb{C}$-vector space such that for all primes p the ring $\mathscr{H}_p(G)$ acts on M and $f \in M$ is a simultaneous eigenform for all $\mathscr{H}_p(G)$ with

$$f|T = \lambda_p(T)f$$

for all $T \in \mathscr{H}_p(G)$. Then $\lambda_p \in \mathrm{Hom}(\mathscr{H}_p(G), \mathbb{C})$. On the other hand let $\widehat{G}(\mathbb{C})_{\mathrm{ss}}$ be the set of conjugacy classes of semisimple elements of $\widehat{G}(\mathbb{C})$. For $c \in \widehat{G}(\mathbb{C})_{\mathrm{ss}}$ and a representation $\rho : \widehat{G}(\mathbb{C}) \to \mathrm{GL}(V_\rho)$ we denote by $\mathrm{trace}(c \mid \rho)$ the trace of $\rho(c)$. Then $\mathrm{trace}(c \mid \cdot)$ extends to a ring homomorphism $\mathrm{tr}(c) : \mathrm{Rep}(\widehat{G}) \to \mathbb{C}$ and the map

$$\mathrm{tr} : \widehat{G}(\mathbb{C})_{\mathrm{ss}} \to \mathrm{Hom}(\mathrm{Rep}(\widehat{G}), \mathbb{C})$$

is a bijection by a classical result due to Chevalley. In particular, we get

Proposition 7.2.1. *The map*

$$\widehat{G}(\mathbb{C})_{\mathrm{ss}} \to \mathrm{Hom}(\mathscr{H}_p(G), \mathbb{C}), \quad c \mapsto \mathrm{tr}(c) \circ \mathrm{Sat}$$

is a bijection.

Using the previous proposition we can now define Satake parameters.

Definition 7.2.2. Let $f \in M$ and $\lambda_p \in \mathrm{Hom}(\mathscr{H}_p(G), \mathbb{C})$ be as above. Let $(c_2, c_3, \ldots)$ be a tuple consisting of elements $c_p \in \widehat{G}(\mathbb{C})_{\mathrm{ss}}$ indexed by the primes such that

$$\mathrm{tr}(c_p) \circ \mathrm{Sat} = \lambda_p$$

for each prime p. Then the elements in $(c_2, c_3, \ldots)$ are called the *Satake parameters of f*.

Let a and a^{-1} be the zeros of the polynomial $X^2 - \tau(p)p^{-11/2}X + 1$ and set

$$\begin{aligned} c_p = \mathrm{diag}(a, a^{-1}) \otimes \mathrm{diag}(p^{-11/2}, p^{-11/2+1}, \ldots, p^{11/2}) \\ \oplus\, \mathrm{diag}(p^{-12}, p^{-11}, \ldots, p^{12}) \in \mathrm{SO}_{49}(\mathbb{C})_{\mathrm{ss}}. \end{aligned}$$

Then the Satake parameters of θ as an eigenform for the Hecke algebra of Sp_{48} are given by

$$(c_2, c_3, \ldots)$$

(see [19, Theorem 6.4]). This allows us to calculate the eigenvalues of θ for all Hecke operators $T \in \mathscr{H}(\mathrm{Sp}_{48}(\mathbb{Z}_p), \mathrm{Sp}_{48}(\mathbb{Q}_p))$. Unfortunately however, $T^{(24)}(p) \notin$

$\mathscr{H}(\mathrm{Sp}_{48}(\mathbb{Z}_p), \mathrm{Sp}_{48}(\mathbb{Q}_p))$. But the action of $\left(T^{(24)}(p)\right)^2$ on the space of Siegel modular forms is (up to normalization) identical to the action of

$$\left(\mathrm{Sp}_{48}(\mathbb{Z}_p)\begin{pmatrix} p^{-1}I_{24} & 0 \\ 0 & I_{24}\end{pmatrix}\mathrm{Sp}_{48}(\mathbb{Z}_p)\right)\cdot\left(\mathrm{Sp}_{48}(\mathbb{Z}_p)\begin{pmatrix} I_{24} & 0 \\ 0 & pI_{24}\end{pmatrix}\mathrm{Sp}_{48}(\mathbb{Z}_p)\right) \in \mathscr{H}(\mathrm{Sp}_{48}(\mathbb{Z}_p), \mathrm{Sp}_{48}(\mathbb{Q}_p))$$

since scalar matrices act trivially. We can therefore compute $\mu(p)^2$ and hence $\mu(p)$ up to sign. Determining the image of $\left(T^{(24)}(p)\right)^2$ under Sat for $\mathscr{H}_p(\mathrm{Sp}_{48})$ is rather tedious and it is easier to work with $T^{(24)}(p)$ directly.

We consider PGSp_{2n}. We have seen that $\mathscr{H}_p(\mathrm{Sp}_{2n}) \hookrightarrow \mathscr{H}_p(\mathrm{PGSp}_{2n})$ and $T^{(n)}(p) \in \mathscr{H}_p(\mathrm{PGSp}_{2n}) \otimes_{\mathbb{Z}} \mathbb{C}$. Recall that the Langlands dual group of PGSp_{2n} is the double cover Spin_{2n+1} of SO_{2n+1} and the covering map $\mathrm{Spin}_{2n+1} \to \mathrm{SO}_{2n+1}$ naturally gives a ring homomorphism $\mathrm{Rep}(\mathrm{SO}_{2n+1}) \hookrightarrow \mathrm{Rep}(\mathrm{Spin}_{2n+1})$. By [58, Section 7, Theorem 4] the diagram

$$\begin{array}{ccc} \mathscr{H}_p(\mathrm{Sp}_{2n}) \otimes_{\mathbb{Z}} \mathbb{C} & \xrightarrow{\mathrm{Sat}} & \mathrm{Rep}(\mathrm{SO}_{2n+1}) \otimes_{\mathbb{Z}} \mathbb{C} \\ \big\downarrow & & \big\downarrow \\ \mathscr{H}_p(\mathrm{PGSp}_{2n}) \otimes_{\mathbb{Z}} \mathbb{C} & \xrightarrow{\mathrm{Sat}} & \mathrm{Rep}(\mathrm{Spin}_{2n+1}) \otimes_{\mathbb{Z}} \mathbb{C} \end{array}$$

commutes. Therefore, the Satake parameters of θ with respect to PGSp_{48} must be in the preimage of $(c_2, c_3, \ldots)$ under the covering map $\mathrm{Spin}_{49} \to \mathrm{SO}_{49}$ and the kernel of the covering map must act non-trivially in $\mathrm{Sat}(T^{(24)}(p))$. In fact by [18, (6.2.8)] (also cf. [37, (3.15)]) we have

$$\mathrm{Sat}(T^{(n)}(p)) = p^{\frac{nk-n(n+1)}{2}} \cdot p^{\frac{n(n+1)}{4}}[\rho_{\mathrm{Spin}}],$$

where ρ_{Spin} is the 2^n-dimensional so-called spin representation of Spin_{2n+1}.

Spin groups and the spin representation

We describe the spin group Spin_m in more detail and construct the spin representation. A nice reference is [32].

Let (V, q) be a quadratic space over a field K. A *Clifford algebra* $\mathrm{CL}(V, \mathrm{q})$ is a pair (B, i), where B is a unital associative K-algebra and i is a linear map $i : V \to B$ that satisfies

$$i(v) \cdot i(v) = \mathrm{q}(v) \cdot 1_B$$

for all $v \in V$ with the following universal property: Every linear map $j : V \to A$ for some unital associative K-algebra A that satisfies $j(v) \cdot j(v) = \mathrm{q}(v) \cdot 1_A$ for all $v \in V$

factors through B, i.e. there exists a unique algebra homomorphism $\varphi : B \to A$ such that

$$\begin{array}{ccc} V & \xrightarrow{i} & B \\ & \searrow^{j} & \downarrow \varphi \\ & & A \end{array}$$

commutes. We can construct $\mathrm{CL}(V, \mathrm{q})$ by

$$B = T(V)/I(V),$$

where $T(V) = K \oplus V \oplus V^{\otimes 2} \oplus \ldots$ is the tensor algebra of V and $I(V)$ is the double-sided ideal generated by all elements of the form $v \otimes v - \mathrm{q}(v)1$. Then i is given by the composition of the natural maps

$$V \hookrightarrow T(V) \to B$$

and is injective. By the universal property of the Clifford algebra, B is unique up to isomorphism. We therefore simply write $\mathrm{CL}(V, \mathrm{q})$ instead of B, drop the i and consider V as a subspace of $\mathrm{CL}(V, \mathrm{q})$.

Now let $V = \mathbb{C}^m$ with basis $(e_i)_{i=1}^m$ and for $v = \sum_{i=1}^m x_i e_i$ set $\mathrm{q}(v) = -\sum_{i=1}^m x_i^2$. Let $\mathrm{CL}_m = \mathrm{CL}(V, \mathrm{q})$. We remark that because all non-degenerate quadratic forms on V are isometric, the choice of q does not matter, however, our choice of q will make the construction of the spin representation easier. The spin group $\mathrm{Spin}_m(\mathbb{C})$ is the multiplicative subgroup of CL_m consisting of elements that are the product of an even number of factors with norm 1, i.e.

$$\mathrm{Spin}_m(\mathbb{C}) = \{v_1 \cdots v_{2k} \mid v_i \in V, q(v_i) = 1 \text{ for } i = 1, \ldots, 2k \text{ and } k \in \mathbb{Z}_{\geq 0}\}.$$

Now let $x, v \in V$. Then

$$\mathrm{q}(x + v)1 = (x + v)^2 = \mathrm{q}(x)1 + \mathrm{q}(v)1 + xv + vx$$

so that

$$xv = -vx + 2(x, v)1,$$

where $(x, v) = \mathrm{q}(x + v)/2 - \mathrm{q}(x)/2 - \mathrm{q}(v)/2$. Hence, if $\mathrm{q}(v) = 1$, then

$$x \mapsto -vxv^{-1} = -vxv = x - 2(x, v)v = \sigma_v(x)$$

defines a reflection on V. Since any element in $\mathrm{SO}_m(\mathbb{C})$ is the product of an even number of reflections, we get a surjective representation

$$\mathrm{Spin}_m(\mathbb{C}) \to \mathrm{SO}_m(\mathbb{C}), \quad \gamma \mapsto (x \mapsto -\gamma x \gamma^{-1}).$$

Elements in the kernel of this map must be in the center of CL_m, which is spanned by 1 if m is even and by 1 and $e_1 \cdots e_m$ if m is odd. The latter element is not in $\mathrm{Spin}_m(\mathbb{C})$ in this case. The involution $\iota : \mathrm{CL}_m \to \mathrm{CL}_m$ defined by $\iota(v_1 \cdots v_k) = v_k \cdots v_1$ gives the identity on multiples of 1 and for $\gamma \in \mathrm{Spin}_m(\mathbb{C})$ we have $\iota(\gamma) = \gamma^{-1}$. Therefore, the kernel of the covering map is $\{\pm 1\}$.

We construct the Spin representation by first constructing a representation of CL_m. With $1 \mapsto (1,1)$ and $e_1 \mapsto (i, -i)$ we see that $\mathrm{CL}_1 \cong \mathbb{C} \oplus \mathbb{C}$. If we interpret $\mathbb{C} \oplus \mathbb{C}$ as the diagonal matrices in $\mathrm{Mat}_2(\mathbb{C})$, we get a representation of CL_1 on $\mathbb{C}^2$. The algebra CL_2 is uniquely determined by the relations $e_1^2 = e_2^2 = -1$ and $e_1 e_2 + e_2 e_1 = 0$. Equally, we have for

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad g_1 g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

that $g_1^2 = g_2^2 = -1$ and $g_1 g_2 + g_2 g_1 = 0$. Therefore, $\mathrm{CL}_2 \cong \mathrm{Mat}_2(\mathbb{C})$. For $m > 2$ we define an isomorphism $\mathrm{CL}_m \xrightarrow{\sim} \mathrm{CL}_{m-2} \otimes \mathrm{Mat}_2(\mathbb{C})$ by

$$e_1 \mapsto 1 \otimes g_1, \quad e_2 \mapsto 1 \otimes g_2, \quad e_j \mapsto (e_{j-2} \otimes i g_1 g_2).$$

Using induction on m we get a representation

$$\rho : \mathrm{CL}_m \xrightarrow{\sim} \begin{cases} \mathrm{Mat}_{2^n}(\mathbb{C}) & \text{if } m = 2n \text{ is even} \\ \mathrm{Mat}_{2^n}(\mathbb{C}) \oplus \mathrm{Mat}_{2^n}(\mathbb{C}) & \text{if } m = 2n+1 \text{ is odd.} \end{cases}$$

Let us from now on consider only the case $m = 2n+1$ is odd. We define the Spin representation on the space $\mathbb{C}^{2^n}$ by

$$\rho_{\mathrm{Spin}} := \pi_1 \circ \rho|_{\mathrm{Spin}_{2n+1}(\mathbb{C})},$$

where $\pi_1 : \mathrm{Mat}_{2^n}(\mathbb{C}) \oplus \mathrm{Mat}_{2^n}(\mathbb{C}) \to \mathrm{Mat}_{2^n}(\mathbb{C})$ is the projection onto the first component. Note that for $j = 1, \dots n$ we have

$$e_{2j-1} \mapsto I_{2^{n-j}} \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^{\otimes (j-1)}$$

$$e_{2j} \mapsto I_{2^{n-j}} \otimes \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^{\otimes (j-1)}$$

7.3 An explicit formula for $r(p)$

We will now construct an element $C_p \in \mathrm{Spin}_{49}(\mathbb{C})$ such that $\pm C_p \mapsto c_p$ under the covering map $\mathrm{Spin}_{49}(\mathbb{C}) \to \mathrm{SO}_{49}(\mathbb{C})$. Since $\rho_{\mathrm{Spin}}(-1) = -I_{2^n}$, we again see that

c_p does not contain the information of the sign of $\mu(p)$. Recall that for isotropic elements $v, v' \in V$ with $(v, v') = 1$ and $a \in \mathbb{C}^\times$ the element $\sigma_{v+v'}\sigma_{v+av'}$ is given by

$$\begin{aligned} v &\mapsto av \\ v' &\mapsto a^{-1}v' \\ w &\mapsto w \quad \text{for } w \in \langle v, v'\rangle^\perp \end{aligned}$$

For $j = 1, \ldots, n$ we define the isotropic elements

$$v_j = \frac{ie_{2j-1} + e_{2j}}{\sqrt{2}}, \quad v_j' = \frac{ie_{2j-1} - e_{2j}}{\sqrt{2}}$$

and

$$\gamma_{j,a} := \frac{v_j + v_j'}{\sqrt{2}} \cdot \frac{v_j + av_j'}{\sqrt{2a}} \in \mathrm{Spin}_{2n+1}(\mathbb{C}),$$

which is mapped to $\sigma_{v_j+v_j'}\sigma_{v_j+av_j'}$ under the covering map $\mathrm{Spin}_{2n+1}(\mathbb{C}) \to \mathrm{SO}_{2n+1}(\mathbb{C})$. Then we get

Lemma 7.3.1. *Let a be a zero of $X^2 - \tau(p)p^{-11/2}X + 1$. Then*

$$C_p := \prod_{j=1}^{12} \gamma_{j,p^{-11/2+j-1}a} \prod_{j=1}^{12} \gamma_{12+j,p^{-12+j-1}}$$

gets mapped to c_p under the covering map $\mathrm{Spin}_{2n+1}(\mathbb{C}) \to \mathrm{SO}_{2n+1}(\mathbb{C})$.

Proof. For the basis

$$(v_1, \ldots, v_{12}, v_{12}', \ldots, v_1', v_{13}, \ldots, v_{24}, e_{2n+1}, v_{24}', \ldots, v_{13}')$$

of V the image of C_p under the covering map takes the form

$$\mathrm{diag}(a, a^{-1}) \otimes \mathrm{diag}(p^{-11/2}, p^{-11/2+1}, \ldots, p^{11/2}) \oplus \mathrm{diag}(p^{-12}, p^{-11}, \ldots, p^{12}).$$

□

Note that

$$\begin{aligned} \gamma_{j,a} &= ie_{2j-1} \cdot \frac{i(1+a)e_{2j-1} + (1-a)e_{2j}}{2\sqrt{a}} \\ &= \frac{1+a}{2\sqrt{a}} \cdot 1 + i\frac{1-a}{2\sqrt{a}} \cdot e_{2j-1}e_{2j} \end{aligned}$$

and so

$$\rho_{\mathrm{Spin}}(\gamma_{j,a}) = \frac{1+a}{2\sqrt{a}} \cdot I_{2^n} + i\frac{1-a}{2\sqrt{a}} \cdot I_{2^{n-j}} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes I_{2^{j-1}}.$$

Since $\operatorname{tr}\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = 0$, only the first summand contributes to the trace of $\rho_{\mathrm{Spin}}(C_p)$ so that

$$\begin{aligned} \operatorname{trace}(C_p \mid \rho_{\mathrm{Spin}}) &= \prod_{j=1}^{12} \left(\frac{1 + p^{-11/2+j-1} a}{2\sqrt{p^{-11/2+j-1} a}} \right) \prod_{j=1}^{12} \left(\frac{1 + p^{-12+j-1}}{2\sqrt{p^{-12+j-1}}} \right) \cdot 2^{24} \\ &= \frac{p^{39}}{a^6} \prod_{j=1}^{12} (1 + p^{-11/2+j-1} a) \prod_{j=1}^{12} (1 + p^{-j}), \end{aligned}$$

where $39 = -\frac{1}{2} \sum_{j=1}^{12} (-11/2 + j - 1) - \frac{1}{2} \sum_{j=1}^{12} (-12 + j - 1)$.

Now we can finally prove

Theorem 7.3.2. *Let p be a prime and let a be a zero of $X^2 - \tau(p) p^{-11/2} X + 1$. Then the representation number $r(p)$ is up to a sign equal to*

$$r(p) = \pm \frac{p^{33}}{a^6} \prod_{j=0}^{11} (1 + p^{-11/2+j} a).$$

Proof. Since $\operatorname{Sat}(T^{(24)}(p)) = p^6 \cdot [\rho_{\mathrm{Spin}}]$, the eigenvalue $\mu(p)$ of θ is

$$\mu(p) = p^6 \cdot \operatorname{trace}(\pm C_p \mid \rho_{\mathrm{Spin}}).$$

We have seen in Theorem 7.1.2 that $r(p) = \mu(p) p^{-12} \cdot \prod_{j=1}^{12} (1 + p^{-j})^{-1}$ so that

$$\begin{aligned} r(p) &= p^{-6} \cdot \operatorname{trace}(\pm C_p \mid \rho_{\mathrm{Spin}}) \prod_{j=1}^{12} (1 + p^{-j})^{-1} \\ &= \pm \frac{p^{33}}{a^6} \prod_{j=0}^{11} (1 + p^{-11/2+j} a). \end{aligned}$$

□

Bibliography

[1] D. Allcock, I. Gal, and A. Mark. The Conway-Sloane calculus for 2-adic lattices. *Enseign. Math.*, 66(1-2):5–31, 2020.

[2] B. C. Berndt, R. J. Evans, and K. S. Williams. *Gauss and Jacobi sums*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons, Inc., New York, 1998. A Wiley-Interscience Publication.

[3] P. Bieker. Invariants for the Weil representation and modular units for orthogonal groups of signature $(2,2)$. *J. Number Theory*, 250:155–182, 2023.

[4] S. Böcherer. Über die Funktionalgleichung automorpher L-Funktionen zur Siegelschen Modulgruppe. *J. Reine Angew. Math.*, 362:146–168, 1985.

[5] S. Böcherer, H. Katsurada, and R. Schulze-Pillot. On the basis problem for Siegel modular forms with level. In *Modular forms on Schiermonnikoog*, pages 13–28. Cambridge Univ. Press, Cambridge, 2008.

[6] R. E. Borcherds. Automorphic forms on $\mathrm{O}_{s+2,2}(\mathbf{R})$ and infinite products. *Invent. Math.*, 120(1):161–213, 1995.

[7] R. E. Borcherds. Automorphic forms with singularities on Grassmannians. *Invent. Math.*, 132(3):491–562, 1998.

[8] R. E. Borcherds. The Gross-Kohnen-Zagier theorem in higher dimensions. *Duke Math. J.*, 97(2):219–233, 1999.

[9] R. E. Borcherds. Reflection groups of Lorentzian lattices. *Duke Math. J.*, 104(2):319–366, 2000.

[10] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. volume 24, pages 235–265. 1997. Computational algebra and number theory (London, 1993).

[11] J. H. Bruinier. *Borcherds products on O(2, l) and Chern classes of Heegner divisors*, volume 1780 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002.

[12] J. H. Bruinier. On the converse theorem for Borcherds products. *J. Algebra*, 397:315–342, 2014.

[13] J. H. Bruinier and E. Freitag. Local Borcherds products. *Ann. Inst. Fourier (Grenoble)*, 51(1):1–26, 2001.

[14] J. H. Bruinier and O. Stein. The Weil representation and Hecke operators for vector valued modular forms. *Math. Z.*, 264(2):249–270, 2010.

[15] J. H. Bruinier, G. van der Geer, G. Harder, and D. Zagier. *The 1-2-3 of modular forms*. Universitext. Springer-Verlag, Berlin, 2008. Lectures from the Summer School on Modular Forms and their Applications held in Nordfjordeid, June 2004.

[16] L. Candelori, C. Franc, and G. S. Kopp. Generating weights for the Weil representation attached to an even order cyclic quadratic module. *J. Number Theory*, 180:474–497, 2017.

[17] J. W. S. Cassels. *Rational quadratic forms*, volume 13 of *London Mathematical Society Monographs*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1978.

[18] G. Chenevier and J. Lannes. *Automorphic forms and even unimodular lattices*, volume 69 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Cham, 2019. Kneser neighbors of Niemeier lattices, Translated from the French by Reinie Erné.

[19] G. Chenevier and O. Taïbi. Siegel modular forms of weight 13 and the Leech lattice. *Enseign. Math.*, 68(1-2):61–98, 2022.

[20] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups*, volume 290 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, third edition, 1999. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov.

[21] F. Diamond and J. Shurman. *A first course in modular forms*, volume 228 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[22] M. Dittmann. Unpublished notes. 2023.

[23] T. Driscoll-Spittler, N. R. Scheithauer, and J. Wilhelm. Reflective modular varieties and their cusps. Preprint, arXiv: 2312.02768, 2023.

[24] S. Ehlen and N.-P. Skoruppa. Computing invariants of the Weil representation. In *L-functions and automorphic forms*, volume 10 of *Contrib. Math. Comput. Sci.*, pages 81–96. Springer, Cham, 2017.

[25] M. Eichler. Quadratische Formen und Modulfunktionen. *Acta Arith.*, 4:217–239, 1958.

[26] M. Eichler. The basis problem for modular forms and the traces of the Hecke operators. In *Modular functions of one variable, I (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, volume 320 of *Lecture Notes in Math.*, pages 75–151. Springer, Berlin-New York, 1973.

[27] M. Eichler. *Quadratische Formen und orthogonale Gruppen*, volume Band 63 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin-New York, 1974. Zweite Auflage.

[28] M. Eichler and D. Zagier. *The theory of Jacobi forms*, volume 55 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1985.

[29] E. Freitag. Die Wirkung von Heckeoperatoren auf Thetareihen mit harmonischen Koeffizienten. *Math. Ann.*, 258(4):419–440, 1981/82.

[30] E. Freitag. *Siegelsche Modulfunktionen*, volume 254 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1983.

[31] E. Freitag. Dimension formulae for vector valued automorphic forms. Preprint, available at `https://www.mathi.uni-heidelberg.de/~freitag/papers.html`, 2012.

[32] T. Friedrich. *Dirac operators in Riemannian geometry*, volume 25 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2000. Translated from the 1997 German original by Andreas Nestke.

[33] J. Gallenkämper. *Themen der Hecke-Theorie zur orthogonalen Gruppe* $O(2, n+2)$. PhD thesis, RWTH Aachen University, 2017. Available at `http://publications.rwth-aachen.de/record/682281/`.

[34] P. B. Garrett. Pullbacks of Eisenstein series; applications. In *Automorphic forms of several variables (Katata, 1983)*, volume 46 of *Progr. Math.*, pages 114–137. Birkhäuser Boston, Boston, MA, 1984.

[35] V. A. Gritsenko. Fourier-Jacobi functions in n variables. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 168:32–44, 187–188, 1988.

[36] V. A. Gritsenko, N.-P. Skoruppa, and D. Zagier. Theta blocks. Preprint, arXiv: 1907.00188, 2019.

[37] B. H. Gross. On the Satake isomorphism. In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 223–237. Cambridge Univ. Press, Cambridge, 1998.

[38] T. Ibukiyama. On differential operators on automorphic forms and invariant pluri-harmonic polynomials. *Comment. Math. Univ. St. Paul.*, 48(1):103–118, 1999.

[39] M. Kashiwara and M. Vergne. On the Segal-Shale-Weil representations and harmonic polynomials. *Invent. Math.*, 44(1):1–47, 1978.

[40] Y. Kitaoka. *Arithmetic of quadratic forms*, volume 106 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1993.

[41] M. Kneser. *Quadratische Formen*. Springer-Verlag, Berlin, 2002. Revised and edited in collaboration with Rudolf Scharlau.

[42] M. Knopp and G. Mason. Vector-valued modular forms and Poincaré series. *Illinois J. Math.*, 48(4):1345–1366, 2004.

[43] A. Krieg, H. Römer, and F. Schaps. Hecke theory for $SO^+(2, n+2)$. *J. Number Theory*, 262:454–470, 2024.

[44] S. S. Kudla and S. Rallis. On the Weil-Siegel formula. *J. Reine Angew. Math.*, 387:1–68, 1988.

[45] R. P. Langlands. *Euler products*, volume 1 of *Yale Mathematical Monographs*. Yale University Press, New Haven, Conn.-London, 1971. A James K. Whittemore Lecture in Mathematics given at Yale University, 1967.

[46] C. Li. From sum of two squares to arithmetic Siegel-Weil formulas. *Bull. Amer. Math. Soc. (N.S.)*, 60(3):327–370, 2023.

[47] H. Maass. Spherical functions and quadratic forms. *J. Indian Math. Soc. (N.S.)*, 20:117–162, 1956.

[48] I. Metzler. *Period integrals of Kudla-Millson lifts and L-functions associated to vector valued modular forms.* PhD thesis, Technische Universität Darmstadt, 2024. Available at `https://tuprints.ulb.tu-darmstadt.de/28711/1/PhD_notes.pdf`.

[49] T. Miyake. *Modular forms.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2006. Translated from the 1976 Japanese original by Yoshitaka Maeda.

[50] M. K.-H. Müller. Invariants of the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ on discriminant forms of odd order. Master's thesis, Technische Universität Darmstadt, 2021.

[51] M. K.-H. Müller. The basis problem for modular forms for the Weil representation. Preprint, arXiv: 2407.01205, 2024.

[52] M. K.-H. Müller. Modular forms for the Weil representation induced from isotropic subgroups. *J. Number Theory*, 263:206–233, 2024.

[53] M. K.-H. Müller and N. R. Scheithauer. The invariants of the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$. To appear in: *Commun. Contemp. Math.* Available at `https://doi.org/10.1142/S0219199725500130`.

[54] G. Nebe, E. M. Rains, and N. J. A. Sloane. *Self-dual codes and invariant theory*, volume 17 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, 2006.

[55] V. V. Nikulin. Integer symmetric bilinear forms and some of their geometric applications. *Math. USSR Izv.*, 14(1):103–167, 1980.

[56] O. T. O'Meara. *Introduction to quadratic forms.* Classics in Mathematics. Springer-Verlag, Berlin, 2000. Reprint of the 1973 edition.

[57] R. A. Rankin. Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions. II. The order of the Fourier coefficients of integral modular forms. *Proc. Cambridge Philos. Soc.*, 35:351–372, 1939.

[58] I. Satake. Theory of spherical functions on reductive algebraic groups over p-adic fields. *Inst. Hautes Études Sci. Publ. Math.*, (18):5–69, 1963.

[59] N. R. Scheithauer. On the classification of automorphic products and generalized Kac-Moody algebras. *Invent. Math.*, 164(3):641–678, 2006.

[60] N. R. Scheithauer. The Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ and some applications. *Int. Math. Res. Not. IMRN*, (8):1488–1545, 2009.

[61] N. R. Scheithauer. Some constructions of modular forms for the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$. *Nagoya Math. J.*, 220:1–43, 2015.

[62] N. R. Scheithauer. Automorphic products of singular weight. *Compos. Math.*, 153(9):1855–1892, 2017.

[63] M. Schwagenscheidt and F. Völz. Lifting newforms to vector-valued modular forms for the Weil representation. *Int. J. Number Theory*, 11(7):2199–2219, 2015.

[64] A. Selberg. On the estimation of Fourier coefficients of modular forms. In *Proc. Sympos. Pure Math., Vol. VIII*, pages 1–15. Amer. Math. Soc., Providence, RI, 1965.

[65] N.-P. Skoruppa. *Über den Zusammenhang zwischen Jacobiformen und Modulformen halbganzen Gewichts*, volume 159 of *Bonner Mathematische Schriften [Bonn Mathematical Publications]*. Universität Bonn, Mathematisches Institut, Bonn, 1985. Dissertation, Rheinische Friedrich-Wilhelms-Universität, Bonn, 1984.

[66] N.-P. Skoruppa. Jacobi forms of critical weight and Weil representations. In *Modular forms on Schiermonnikoog*, pages 239–266. Cambridge Univ. Press, Cambridge, 2008.

[67] O. Stein. On analytic properties of the standard zeta function attached to a vector-valued modular form. *Res. Number Theory*, 8(4):Paper No. 86, 27, 2022.

[68] J.-L. Waldspurger. Engendrement par des séries thêta de certains espaces de formes modulaires. *Invent. Math.*, 50(2):135–168, 1978/79.

[69] A. Weil. Sur certains groupes d'opérateurs unitaires. *Acta Math.*, 111:143–211, 1964.

[70] A. Weil. Sur la formule de Siegel dans la théorie des groupes classiques. *Acta Math.*, 113:1–87, 1965.

[71] F. Werner. *Vector valued Hecke theory*. PhD thesis, Technische Universität Darmstadt, 2014. Available at `https://tuprints.ulb.tu-darmstadt.de/4238/1/phd.pdf`.

[72] D. Zagier. Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields. In *Modular functions of one variable, VI (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976)*, volume 627 of *Lecture Notes in Math.*, pages 105–169. Springer, Berlin-New York, 1977.

[73] S. Zemel. Integral bases and invariant vectors for Weil representations. *Res. Number Theory*, 9(1):Paper No. 5, 27, 2023.

[74] W. Zhang. *Modularity of Generating Functions of Special Cycles on Shimura Varieties*. PhD thesis, Columbia University, 2004. Available at `https://www.math.columbia.edu/~thaddeus/theses/2009/zhang.pdf`.

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